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Experimentally feasible measures of distance between quantum operations

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Abstract We present two measures of distance between quantum processes which can be measured directly in laboratory without resorting to process tomography. The measures are based on the superfidelity, introduced recently to provide an upper bound for quantum fidelity. We show that the introduced measures partially fulfill the requirements for distance measure between quantum processes. We also argue that they can be especially useful as diagnostic measures to get preliminary knowledge about imperfections in an experimental setup. In particular we provide quantum circuit which can be used to measure the superfidelity between quantum processes. We also provide a physical interpretation of the introduced metrics based on the continuity of channel capacity.

Keywords Quantum information · Quantum noise · Quantum tomography

1 Introduction

Recent applications of quantum mechanics are based on processing and transferring information encoded in quantum states [1,2]. The full description of quantum information processing procedures is given in terms of *quantum channels* or *quantum processes*, *ie.* completely positive, trace non-increasing maps on the set of quantum states [1].

In many areas of quantum information processing one needs to quantify the difference between ideal quantum procedure and the procedure which is performed in the laboratory. Theoretically these imperfections can be measured using state tomog-

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raphy [3,4] or process tomography [5,6]. In particular the problem of quantifying the distance between quantum channels was studied in the context of channel distinguishability [7–10]. The problem of discrimination of memory-channels was considered in Ref. [11], where it was shown that memory assisted protocols are needed in this scenario.

The problem of identifying a universal measure which could be used for this purpose was first comprehensively addressed in Ref. [12]. In this work the authors provided the list of requirements which should be satisfied theoretically, as well as experimentally, in order to make the measures of distance between quantum processes meaningful.

Another approach to define a fidelity criterion for quatum channels is presented in Ref. [13]. Authors define *minimax fidelity* which is noncommutative generalization of maximal Hellinger distance between classical positive probability kernels. The *minimax fidelity* has a direct operational meaning and it gives free dimensional bounds to the CB-norm distance. Unfortunately to obtain *minimax fidelity* one must perform optimization procedure with respect to the set of quantum states.

The main aim of this paper is to present two measures of distance between quantum processes which can be measured directly in laboratory without resorting to process tomography. For this purpose we use measures based on the *superfidelity*, the functional introduced recently [14–16], to provide an upper bound for quantum fidelity. We introduce metrics on the space of quantum operations based on superfidelity and we examine their properties. We propose a simple quantum circuit which allows for the measurement of superfidelity between quantum processes. Hence, to our knowledge, we provide the first examples of metrics on the space of quantum operations which can be measured directly in laboratory without resorting to process tomography. We test our quantities against the requirements introduced in Ref. [12] and show their relations with J fidelity introduced therein. We provide a physical interpretation of the introduced metrics based on the continuity of channel capacity. We also argue that the proposed metrics can be especially useful as the diagnostic measures allowing to get preliminary knowledge about imperfections in an experimental setup.

2 Preliminaries

Let $\mathcal{H}$ be a separable, complex Hilbert space used to describe the system in question. The state of the system is described by the density matrix, *ie.* operator $\rho : \mathcal{H} \to \mathcal{H}$, which is positive ($\rho \geq 0$) and normalized ($\mathrm{tr}\rho = 1$).

In what follows we denote by $\mathcal{M}_N$ the space of density matrices of size N. We restrict our attention to the finite-dimensional case.

Recently a new measure of similarity between quantum states, namely *superfidelity* $G(\rho, \sigma)$, was introduced [14]

$$G(\rho, \sigma) = \mathrm{tr}\rho\sigma + \sqrt{1 - \mathrm{tr}\rho^2}\sqrt{1 - \mathrm{tr}\sigma^2}. \tag{1}$$

The most interesting feature of the superfidelity is that it provides an upper bound for quantum fidelity [14]

$$F(\rho_1, \rho_2) \leq G(\rho_1, \rho_2). \tag{2}$$

It also provides lower [16] and upper [15] bound for the trace distance

$$1 - G(\rho_1, \rho_2) \leq D_{\mathrm{tr}}(\rho_1, \rho_2) \leq \sqrt{\frac{\tau}{2}}\sqrt{1 - G(\rho_1, \rho_2)}, \tag{3}$$

where $\tau = \mathrm{rank}(\rho_1 - \rho_2)$. In (2) we have an equality either for $\rho, \sigma \in \mathcal{M}_2$ or in the case where one of the states is pure.

The superfidelity has also properties which make it useful for quantifying the distance between quantum states. In particular we have:

1. Bounds: $0 \leq G(\rho_1, \rho_2) \leq 1$.
2. Symmetry: $G(\rho_1, \rho_2) = G(\rho_2, \rho_1)$.
3. Unitary invariance: for any unitary operator U, we have

$$G(\rho_1, \rho_2) = G(U\rho_1 U^\dagger, U\rho_2 U^\dagger).$$

4. Concavity: $G(\rho_1, \alpha\rho_2 + (1-\alpha)\rho_3) \geq \alpha G(\rho_1, \rho_2) + (1-\alpha) G(\rho_1, \rho_3)$ for any $\rho_1, \rho_2, \rho_3 \in \mathcal{M}_N$ and $\alpha \in [0, 1]$.
5. Supermultiplicativity: for $\rho_1, \rho_2, \rho_3, \rho_4 \in \mathcal{M}_N$ we have

$$G(\rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4) \geq G(\rho_1, \rho_3) G(\rho_2, \rho_4).$$

Note that the superfidelity shares properties 1.-4. with fidelity. However, in contrast to the fidelity, the superfidelity is not multiplicative, but supermultiplicative.

In Ref. [15] the authors showed that G is *jointly* concave in its two arguments. Note that the property of joint concavity is obeyed by square root of the fidelity but not by the fidelity.

It was also shown that it can be used to define such metrics on $\mathcal{M}_N$ [14] as

$$C_G(\rho, \sigma) = \sqrt{1 - G(\rho, \sigma)} \tag{4}$$

or

$$A_{G^2}(\rho_1, \rho_2) = \arccos(G(\rho_1, \rho_2)). \tag{5}$$

The problem of finding the measure of difference between ideal and real quantum processes was first studied in depth in Ref. [12], where the authors proposed the list of requirements for gold-standard metric between quantum processes.

If Δ is a candidate for distance measure, the criteria are as follows:

(R1) *Metric*: Δ should be a metric.
(R2) *Easy to calculate*: it should be possible to evaluate Δ in a direct manner.
(R3) *Easy to measure*: there should be a clear and achievable experimental procedure for determining the value of Δ.

(R4) *Physical interpretation*: Δ should have a well-motivated physical interpretation.
(R5) *Stability*: $\Delta(\mathbb{1} \otimes \Phi, \mathbb{1} \otimes \Psi) = \Delta(\Phi, \Psi)$, where $\mathbb{1}$ is the identity operation on an additional quantum system.
(R6) *Chaining*: $\Delta(\Phi_2 \circ \Phi_1, \Psi_2 \circ \Psi_1) \leq \Delta(\Phi_1, \Psi_1) + \Delta(\Phi_2, \Psi_2)$.

As already noted in Ref. [12], it is hard to find a quantity which fulfills all of the above requirements. On contrary, in many cases it is desirable to use some kind of quantity which does not posses all of the required features to get some preliminary insight into the nature of errors occurring in the experimental setup.

3 Metrics based on superfidelity

Let Δ_G be a distance measure based on the superfidelity between Jamiołkowski states of processes [1]. In this paper we consider two functions C_G, motivated by root infidelity,

$$C_G(\Phi, \Psi) = \sqrt{1 - G(\rho_\Phi, \rho_\Psi)}, \tag{6}$$

and A_{G^2}, motivated by Bures angle,

$$A_{G^2}(\Phi, \Psi) = \arccos G(\rho_\Phi, \rho_\Psi). \tag{7}$$

Note that the above metrics defines the same topology.

Superfidelity is a continuous function on the space of quantum states with respect to both arguments. Thus, metrics C_G and A_{G^2} are continuous with respect to a perturbation of a given map.

3.1 Basic properties (R1, R2)

It was shown in Ref. [14] that quantities defined in Eqs. (4) and (5) do provide the metrics on the space of quantum states. As such C_G and A_{G^2} fulfill requirement (R1).

Also from the definition of superfidelity it is clear that Δ_G can be easily calculated—requirement (R2). From the computational point of view the calculation of Δ_G, using standard mathematical software, is also much efficient than in the case of metrics based on fidelity [15].

3.2 Measurement procedure (R3)

Any useful distance measure for quantum processes should be easy to measure in a laboratory. In the case of any metric based on superfidelity this is to say that it should be easy to measure the superfidelity between quantum processes.

In Fig. 1 a quantum circuit used for measuring the superfidelity between two quantum processes is presented. In the first step one needs to produce Jamiołkowski matri-

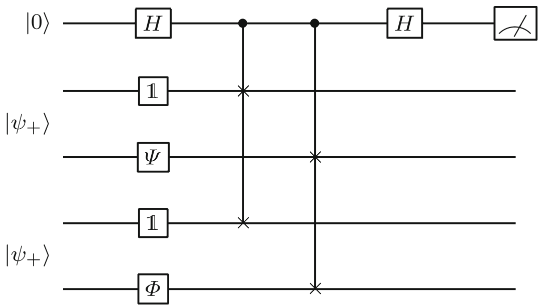

Fig. 1 Quantum circuit for measuring $\mathrm{tr}\rho_\Phi\rho_\Psi$. The probability P_0 of finding the top qubit in state $|0\rangle$ leads to an estimation of $\mathrm{tr}\rho_\Phi\rho_\Psi = 2P_0 - 1$ [17]. This allows direct estimation of process superfidelity. See Ref. [18] for the description of quantum gates used in this circuit

ces for analyzed processes as described in Ref. [6]. In the second step we utilize the scheme proposed in Ref. [17].

The circuit works for quantum channels of an arbitrary dimension. Its only drawback is that it requires controlled SWAP operation, which makes it problematic for realization using contemporary technology [19].

In order to measure the superfidelity between two one-qubit channels one needs five qubits and for measuring the superfidelity between two n-dimensional states one needs $2 + 4n$ dimensional space. Note that the measurement is performed on a qubit indeferently of the dimensionality of the system on which the channels act.

One should also note that the presented quantum circuit can be used to measure the fidelity between a unitary operation and an arbitrary quantum channel. As such it can be used in the situation when one needs to measure the difference between an ideal (*ie.* unitary) process and a real (*ie.* noisy) process.

The advantage of the presented measurement procedure is that it requires less resources than process tomography of channels one wish to compare. One can compare the method presented above with ancilla-assisted process tomography (AAPT) [20]. In order to perform full AAPT of a quantum process described by Jamiołkowski matrix of dimension $d = 2n$ one needs to perform $d + 1$ measurements using mutual unbiased bases. In our case one needs only one one-qubit measurement.

3.3 Physical interpretation (R4)

The physical interpretation of superfidelity based metrics can be given in terms of continuity of channels capacities. The results in Ref. [21] give the connection of capacities with the diamond norm, defined for mappings from $\mathcal{B}(\mathcal{H}_{\mathrm{in}})$ to $\mathcal{B}(\mathcal{H}_{\mathrm{out}})$ as

$$\|\Phi\|_\diamond = \max\{\|(\Phi \otimes \mathbb{1})(X)\|_1 : X \in \mathcal{B}(\mathcal{H}_{\mathrm{in}} \otimes \mathcal{H}_{\mathrm{ref}}), \|X\|_1 = 1\}. \tag{8}$$

Using stability results concerning diamond norm [22, Chap. 11] one can take $\dim(\mathcal{H}_{\mathrm{ref}}) = \dim(\mathcal{H}_{\mathrm{in}})$.

To obtain continuity first note that the inequality (3) gives an upper bound for the trace distance which depends on system dimension in terms of C_G. From the equivalence of topology generated by A_{G^2} and C_G, it follows that for a fixed dimension if two channels are close in the superfidelity based metrics they are close in trace distance defined for Jamiołkowski matrices. All norms on finite dimensional space are equivalent. Taking into account the stability results concerning diamond norm, one can see that trace norm defined for Jamiołkowski matrices and diamond norm defines the same topology.

Thus if two channels are close in the superfidelity based metrics they are close in metric generated by diamond norm [23]. Following results from Ref. [21] we obtain that two channels close in the superfidelity based metrics have similar classical capacity, quantum capacity, and private classical capacity.

3.4 Stability (R5)

In this paragraph we show the *stability* of distance measures based on superfidelity Δ_G between Jamiołkowski matrices of processes. In fact we will even show, that if we extend both channels by the same unitary channel (not necessarily identity) the superfidelity-based distance measures do not change.

We have the following lemma.

Lemma 1 *Let Ψ, Φ be given channels and let τ be a unitary quantum channel, then*

$$G(\rho_{\tau\otimes\Psi}, \rho_{\tau\otimes\Phi}) = G(\rho_\Psi, \rho_\Phi). \tag{9}$$

Proof To prove the above all we need is the fact that Jamiołkowski state of unitary channel is a rank 1 projector, the fact that $\rho_{\Psi_1\otimes\Psi_2}$ is a permutation similar to $\rho_{\Psi_1} \otimes \rho_{\Psi_2}$ and the following lemma. □

Lemma 2 *Let $|\phi\rangle$ be a normalized vector, then*

$$G(|\phi\rangle\langle\phi| \otimes \rho_1, |\phi\rangle\langle\phi| \otimes \rho_2) = G(\rho_1, \rho_2). \tag{10}$$

Proof To obtain the lemma it is enough to notice that

$$\mathrm{tr}(|\phi\rangle\langle\phi| \otimes \rho_i)(|\phi\rangle\langle\phi| \otimes \rho_j) = \mathrm{tr}(|\phi\rangle\langle\phi|\,|\phi\rangle\langle\phi|)\mathrm{tr}\rho_i\rho_j = \mathrm{tr}\rho_i\rho_j \tag{11}$$

for any $i, j \in \{1, 2\}$. □

From Lemma 1 we have that any Δ_G fulfills requirement (R5).

3.5 Chaining (R6)

Despite its simple form superfidelity, in contrast to fidelity or trace distance, is not monotone under the action of quantum channels. This fact was proved in Ref. [15]. One can easily construct an example similar to the one used in Ref. [15] to see that the superfidelity between quantum channels fails to fulfill requirement (R6).

To get counterexample one may consider the following Jamiołkowski states

$$\rho_{\Phi_1} = \rho_{\Phi_2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{12}$$

$$\rho_{\Psi_1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho_{\Psi_2} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{13}$$

representing quantum channels Φ_1, Φ_2, Ψ_1 and Ψ_2 respectively. However, this property holds if we aim to compare unitary (*ie.* ideal) quantum operations with general (*ie.* noisy) quantum operations. In this particular case superfidelity reduces to J fidelity.

Chaining rule is important if one aims to compare quantum processes divided into smaller steps. It holds for distance measures proposed in Ref. [12].

4 Examples

To get a deeper insight into a behavior of superfidelity-based distances we provide explicit formulas for the selected families of quantum channels.

4.1 One-qubit channels

We start by analyzing one-qubit channels. In this case dynamical matrix can parametrized as Ref. [24] (up to two orthogonal transformations [1, Sect. 10.7])

$$D = \frac{1}{2}\begin{pmatrix} \eta_z + \kappa_z + 1 & 0 & \kappa_x + i\kappa_y & \eta_x + \eta_y \\ 0 & -\eta_z + \kappa_z + 1 & \eta_x - \eta_y & \kappa_x + i\kappa_y \\ \kappa_x - i\kappa_y & \eta_x - \eta_y & -\eta_z - \kappa_z + 1 & 0 \\ \eta_x + \eta_y & \kappa_x - i\kappa_y & 0 & \eta_z - \kappa_z + 1 \end{pmatrix}, \tag{14}$$

where parameters $\boldsymbol{\kappa} = (\kappa_x, \kappa_y, \kappa_z)$ and $\boldsymbol{\eta} = (\eta_x, \eta_y, \eta_z)$ are real vectors representing distortion and translation of the quantum state in the Bloch ball.

Let D_Ψ and D_Φ be two dynamical matrices parametrized by vectors $\boldsymbol{\kappa}_\Psi$, $\boldsymbol{\eta}_\Psi$ and $\boldsymbol{\kappa}_\Phi$, $\boldsymbol{\eta}_\Phi$ respectively.

After straightforward calculations we get

$$G(\rho_\Psi, \rho_\Phi) = \frac{1}{4}\Big(1 + \boldsymbol{\kappa}_\Psi \cdot \boldsymbol{\kappa}_\Phi + \boldsymbol{\eta}_\Psi \cdot \boldsymbol{\eta}_\Phi + \sqrt{3 - ||\boldsymbol{\kappa}_\Psi||^2 - ||\boldsymbol{\eta}_\Psi||^2}\sqrt{3 - ||\boldsymbol{\kappa}_\Phi||^2 - ||\boldsymbol{\eta}_\Phi||^2}\Big), \tag{15}$$

where '·' denotes the scalar product.

One should note that it is hard to obtain concise formula for the fidelity or trace distance between two one-qubit channels.

4.2 Selected higher-dimensional channels

We start with an elementary result concerning the superfidelity on commuting matrices [14].

Lemma 3 *Let ρ_1 and ρ_2 be hermitian matrices with eigenvalues $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ respectively. If $\rho_1\rho_2 = \rho_2\rho_1$ then there exists an orthonormal basis $\{|i\rangle\}_i$ such that*

$$\rho_1 = \sum_i \lambda_i \, |i\rangle \, \langle i| \quad \text{and} \quad \rho_2 = \sum_i \mu_i \, |i\rangle \, \langle i| \, . \tag{16}$$

With this notation we have

$$G(\rho_1, \rho_2) = \boldsymbol{\lambda} \cdot \boldsymbol{\mu} + \sqrt{(1 - |\boldsymbol{\lambda}|^2)(1 - |\boldsymbol{\mu}|^2)}. \tag{17}$$

This lemma enables us to obtain explicit formulas for the superfidelity between quantum channels for some interesting families discussed below.

4.2.1 Depolarizing channel

For any $p \in [0, 1]$ we define a depolarizing channel as Ref. [2]

$$\kappa_{d,p}(\rho) = p\rho + (1 - p)\mathrm{tr}(\rho)\frac{1}{d}\mathbb{1}. \tag{18}$$

It is a d-dimensional CP-TP map. It is not difficult to notice that $\rho_{\kappa_{d,p}}$ and $\rho_{\kappa_{d,q}}$ commute, thus we have

$$G(\rho_{\kappa_{d,p}}, \rho_{\kappa_{d,q}}) = \frac{1}{d^2}\left(1 + (d^2 - 1)pq + (d^2 - 1)\sqrt{(1 - p^2)(1 - q^2)}\right). \tag{19}$$

4.2.2 Generalized Pauli channel

Generalized Pauli channel Π_d is an extension to any dimension of the one-qubit Pauli channel [2].

For two generalized Pauli channels ρ_p and ρ_q given by the probability distribution matrices $p_{i,j}$ and $q_{i,j}$, we can find a direct formula for their similarity in terms of superfidelity

$$G(\rho_p, \rho_q) = \mathrm{tr}(pq^T) + \sqrt{1 - \mathrm{tr}(pp^T)}\sqrt{1 - \mathrm{tr}(qq^T)}. \tag{20}$$

This follows from the fact that ρ_p and ρ_q commute and vectors $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{d^2}$ are eigenvalues of ρ_p and ρ_q respectively.

4.2.3 Werner-Holevo channel

Werner-Holevo channel cannot be represented as generalized Pauli channel.

For dimension d and parameter $p \in [-\frac{1}{d-1}, \frac{1}{d+1}]$ we define Werner-Holevo channel as

$$\kappa^T_{d,p}(\rho) = p\rho^T + (1-p)\mathrm{tr}(\rho)\frac{1}{d}\mathbb{1}. \tag{21}$$

Thus we have

$$G(\rho^T_{\kappa_{d,p}}, \rho^T_{\kappa_{d,q}}) = \frac{1}{d^2}\left(1 + (d^2-1)pq + (d^2-1)\sqrt{(1-p^2)(1-q^2)}\right). \tag{22}$$

Since the dynamical matrices for depolarizing channel and Werner-Holevo channel commute, one can also easily calculate the superfidelity between these channels. In this case it reads

$$G(\rho_{\kappa_{d,p}}, \rho^T_{\kappa_{d,q}}) = \frac{1}{d^2}\left(1 + (d-1)pq + (d^2-1)\sqrt{(1-p^2)(1-q^2)}\right). \tag{23}$$

4.2.4 Dephasing channel

Let $F_t = F_t^\dagger$ be a d-dimensional *dephasing matrix ie.* $(F_t)_{ii} = 1$ and $(F_t)_{ij} = f_{ij}(t)$ for $i \neq j$. We define channel D_{F_t} as follows

$$D_{F_t} : \rho_0 \to F_t \bullet \rho_0, \tag{24}$$

where by '$\bullet$' we denoted the Hadamard product of matrices. One can easily see that for these types of channels

$$G_t(\mathbf{f}, \mathbf{g}) = \frac{1}{d^2}\left((\mathbf{f}_t)^\dagger \cdot \mathbf{g}_t + \sqrt{d^2 - \|\mathbf{f}_t\|^2}\sqrt{d^2 - \|\mathbf{g}_t\|^2},\right) \tag{25}$$

which in the case $\mathbf{g}_t = \mathbf{f}_t^*$ reduces to

$$G_t(\mathbf{f}, \mathbf{f}^*) = 1 - \frac{\|f_t\|^2 - (\mathbf{f}_t)^\dagger \cdot \mathbf{f}_t^*}{d^2}. \tag{26}$$

Here $\mathbf{f}_t$ stands for a vector obtained from matrix F_t by the reshaping procedure [1], and $\|\cdot\|$ represents a standard norm on $\mathbb{C}^d$.

Note also that the results (25) and (26) hold in the case of arbitrary hermitian matrix F_t, not only a dephasing matrix.

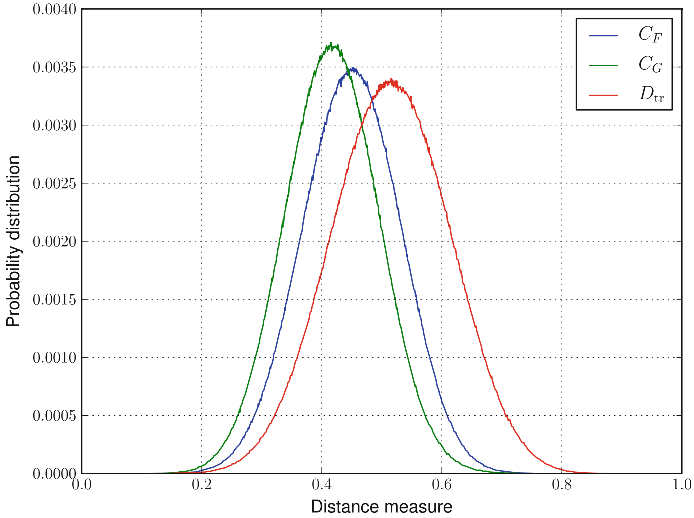

Fig. 2 Probability distributions of trace distance (D_{tr}), root infidelity (C_F) and root "superinfidelity" ($C_G = \sqrt{1-G}$) for one-qubit quantum channels

4.3 Statistical properties

In order to asses the quality of the distance measures based on superfidelity we have analyzed its statistical behavior. We have compared the average superfidelity with the average fidelity between one-qubit quantum channels. We have also analyzed average superfidelity and average fidelity between quantum channels for higher-dimensional random channels.

Measures based on fidelity and trace distance (J fidelity and J process distance) provide natural benchmarks for testing new measures on the space of quantum operations.

Using the algorithm by Bruzda et al.[25] we have generated 10^6 pairs of normalized dynamical matrices representing one-qubit quantum channels. For this sample we have calculated the distance measures C_F, C_g, D_{tr} (see Fig. 2).

Numerical results presented in Fig. 2 indicate that in the case of one-qubit channels the superfidelity (or metrics based on it) can be used to approximate trace distance or measures based on fidelity. Thus, the circuit used to measure the superfidelity can be used to provide some insight into the behavior of these measures.

5 Concluding remarks

We have introduced the measure of similarity between quantum processes constructed as the superfidelity between corresponding Jamiołkowski states. We have used this quantity to introduce two metrics on the space of quantum operations—C_G and A_G—motivated by root infidelity and Bures angle. We have argued that the introduced quantities can be used as diagnostic measures for probing errors occurring during physical realizations of quantum information processing. This is especially true as we have shown that the presented quantities can be potentially measured in laboratory. Also, a quantum circuit, constructed to measure the superfidelity can be used to measure the fidelity between a unitary evolution, regarded as an ideal channel, and an arbitrary quantum process, realized in a laboratory. Thus, the presented quantum circuit can be used to calibrate experimental setup with respect to some ideal setup. We also provide a physical interpretation of the introduced metrics based on the continuity of channel capacity. For the special case of one-qubit channels superfidelity between quantum operations can be used as a relatively good approximation of the fidelity.

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