

Bound on trace distance based on superfidelity

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We provide a bound for the trace distance between two quantum states. The lower bound is based on the superfidelity, which provides the upper bound on quantum fidelity. One of the advantages of the presented bound is that it can be estimated using a simple measurement procedure. We also compare this bound with the one provided in terms of fidelity.

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I. INTRODUCTION

The trace distance is one of the most natural distance measures used in quantum-information theory [1,2]. It is one of the main tools used in distinguishability theory and it is connected to the average success probability when distinguishing two states by a measurement [3–5]. It is also related to quantum fidelity, which provides the measure of similarity of two quantum states [1,6,7].

Both quantities are particularly important in quantum cryptography, since the security of quantum protocols relies on an ability to measure the distance between two quantum states [4]. The trace distance is also related to other properties of quantum states like the von Neumann entropy and relative entropy [1].

The main aim of this work is to provide a lower bound for the trace distance using measurable quantities. We use nonlinear functions of the form $\text{tr} \rho_i \rho_j$, $i, j = 1, 2$, where ρ_i and ρ_j are density matrices. For such forms there exist feasible schemes to measure them in an experiment without resorting to state tomography [8]. We give a lower bound based on the superfidelity introduced recently in [9]. Proof of this bound gives an answer to the conjecture stated by Mendonca *et al.* in [10].

II. BOUNDS ON TRACE DISTANCE

Let us denote by Ω_N the space of density matrices acting on N -dimensional Hilbert space \mathbb{C}^N . For two density matrices $\rho_1, \rho_2 \in \Omega_N$ the trace distance is defined as

$$D_{\text{tr}}(\rho_1, \rho_2) = \frac{1}{2} \text{tr} |\rho_1 - \rho_2|. \quad (1)$$

In the particular case of pure states we can use Bloch vectors $\rho_1 = \vec{r} \cdot \vec{\sigma}$ and $\rho_2 = \vec{s} \cdot \vec{\sigma}$. One can see that the trace distance between such states is equal to half of the Euclidean distance between the respective Bloch vectors,

$$D_{\text{tr}}(\rho_1, \rho_2) = \frac{|\vec{r} - \vec{s}|}{2}. \quad (2)$$

The trace distance can be bounded with the use of the fidelity [1,4],

$$1 - \sqrt{F(\rho_1, \rho_2)} \leq D_{\text{tr}}(\rho_1, \rho_2) \leq \sqrt{1 - F(\rho_1, \rho_2)}, \quad (3)$$

where the fidelity is defined as

$$F(\rho_1, \rho_2) = (\text{tr} |\sqrt{\rho_1} \sqrt{\rho_2}|)^2. \quad (4)$$

The inequality (3) shows that F and D_{tr} are closely related indicators of distinguishability.

The main result of this work is a lower bound for the trace distance, which we prove in the next section,

$$1 - G(\rho_1, \rho_2) \leq D_{\text{tr}}(\rho_1, \rho_2), \quad (5)$$

where G is called the *superfidelity*, and was introduced in [9]. For $\rho_1, \rho_2 \in \Omega_N$ it is defined as

$$G(\rho_1, \rho_2) = \text{tr} \rho_1 \rho_2 + \sqrt{1 - \text{tr} \rho_1^2} \sqrt{1 - \text{tr} \rho_2^2}. \quad (6)$$

From the matrix analytic perspective the inequality (5) relates the trace norm on the space Ω_N to the Hilbert-Schmidt scalar product on Ω_N .

For the sake of consistency we provide basic information about the superfidelity [9]. The most interesting feature of superfidelity is that it provides an upper bound for quantum fidelity [9]

$$F(\rho_1, \rho_2) \leq G(\rho_1, \rho_2). \quad (7)$$

The superfidelity also has properties which make it useful for quantifying the distance between quantum states. In particular, we have (1) Bounds: $0 \leq G(\rho_1, \rho_2) \leq 1$. (2) Symmetry: $G(\rho_1, \rho_2) = G(\rho_2, \rho_1)$. (3) Unitary invariance: for any unitary operator U , we have $G(\rho_1, \rho_2) = G(U\rho_1 U^\dagger, U\rho_2 U^\dagger)$. (4) Concavity: $G(\rho_1, \alpha\rho_2 + (1-\alpha)\rho_3) \geq \alpha G(\rho_1, \rho_2) + (1-\alpha)G(\rho_1, \rho_3)$ for any $\rho_1, \rho_2, \rho_3 \in \Omega_N$ and $\alpha \in [0, 1]$. (5) Supermultiplicativity: for $\rho_1, \rho_2, \rho_3, \rho_4 \in \Omega_N$ we have

$$G(\rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4) \geq G(\rho_1, \rho_3)G(\rho_2, \rho_4). \quad (8)$$

Note that the superfidelity shares properties 1–4 with the fidelity. However, in contrast to the fidelity, the superfidelity is not multiplicative, but supermultiplicative.

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In [10] the authors showed that G is *jointly* concave in its two arguments,

$$\begin{aligned} G(\alpha\rho_1 + (1-\alpha)\rho_2, \alpha\rho'_1 + (1-\alpha)\rho'_2) \\ \geq \alpha G(\rho_1, \rho'_1) + (1-\alpha)G(\rho_2, \rho'_2), \end{aligned} \quad (9)$$

for $\alpha \in [0, 1]$. Note that the property of joint concavity is obeyed by the square root of fidelity $\sqrt{F(\rho_1, \rho_2)} = \text{tr}[\sqrt{\rho_2}\sqrt{\rho_1}]$, but not by the fidelity (4).

Fidelity can be used to define the metric $D_B(\rho_1, \rho_2)$ on the space Ω_N as

$$D_B(\rho_1, \rho_2) = \sqrt{2 - 2\sqrt{F(\rho_1, \rho_2)}}. \quad (10)$$

Unfortunately the analog of the Bures distance defined using the superfidelity

$$D'_B(\rho_1, \rho_2) = \sqrt{2 - 2\sqrt{G(\rho_1, \rho_2)}} \quad (11)$$

is not a metric [10], but the quantity

$$D_G(\rho_1, \rho_2) = \sqrt{2 - 2G(\rho_1, \rho_2)} \quad (12)$$

provides the metric on the space Ω_N [9,10].

Finally one should note that the superfidelity is particularly convenient to use as the practical measure of similarity between quantum states. One of the main advantages of superfidelity is that it is possible to design feasible schemes to measure it in an experiment [9]. Also, from the computational point of view, calculation of the superfidelity is significantly less resource consuming [10].

III. MAIN RESULT

The properties of superfidelity listed above suggest that it would be convenient to use it instead of fidelity to draw conclusions about the distinguishability of quantum states. This section shows how this can be done by relating the superfidelity and trace distance.

First we can observe that from the inequality

$$F(\rho_1, \rho_2) \leq G(\rho_1, \rho_2), \quad (13)$$

and since we have (3), we get that

$$1 - \sqrt{G(\rho_1, \rho_2)} \leq D_{\text{tr}}(\rho_1, \rho_2). \quad (14)$$

Our main aim is to prove the following inequality, which provides a tighter bound.

Theorem 1. For any $\rho_1, \rho_2 \in \Omega_N$ we have

$$1 - G(\rho_1, \rho_2) \leq D_{\text{tr}}(\rho_1, \rho_2), \quad (15)$$

or equivalently

$$\frac{1}{2} \text{tr}|\rho_1 - \rho_2| + \text{tr} \rho_1 \rho_2 + \sqrt{1 - \text{tr} \rho_1^2} \sqrt{1 - \text{tr} \rho_2^2} \geq 1. \quad (16)$$

This inequality was first stated as a conjecture in [10], where it was verified numerically for small dimensions. Clearly it is motivated by the lower bound for trace distance provided by the inequality (3).

To prove Theorem 1 we need the following lemma.

Lemma 1. For any $\rho_1, \rho_2 \in \Omega_N$ let P_+ and P_- be the pro-

jectors onto $(\rho_1 - \rho_2)^+$ and $(\rho_1 - \rho_2)^-$, respectively. We have the following inequalities:

$$\text{tr} P_+(1 - \rho_1)\rho_1 \geq \text{tr} P_+(1 - \rho_1)\rho_2, \quad (17)$$

$$\text{tr} P_-\rho_1(1 - \rho_1) \geq \text{tr} P_-\rho_1(1 - \rho_2), \quad (18)$$

$$\text{tr} P_+(1 - \rho_2)\rho_2 \geq \text{tr} P_+(1 - \rho_1)\rho_2, \quad (19)$$

$$\text{tr} P_-\rho_2(1 - \rho_2) \geq \text{tr} P_-\rho_1(1 - \rho_2). \quad (20)$$

Proof. Because of the similarity we will show only inequality (17). It is easy to prove inequalities (18)–(20) in a similar manner.

We subtract the right- from the left-hand side of (17) to get

$$\begin{aligned} \text{tr} P_+(1 - \rho_1)\rho_1 - \text{tr} P_+(1 - \rho_1)\rho_2 \\ = \text{tr} P_+(1 - \rho_1)(\rho_1 - \rho_2) \\ = \text{tr} P_+(1 - \rho_1)(\rho_1 - \rho_2)P_+ \geq 0 \end{aligned} \quad (21)$$

because $1 - \rho_1$ is positive semidefinite. ■

Proof of Theorem 1. Adding the inequalities (17) and (18) we get

$$\text{tr} \rho_1(1 - \rho_1) \geq \text{tr} P_+(1 - \rho_1)\rho_2 + \text{tr} P_-\rho_1(1 - \rho_2). \quad (22)$$

Similarly, by adding inequalities (19) and (20) we get

$$\text{tr} \rho_2(1 - \rho_2) \geq \text{tr} P_+(1 - \rho_1)\rho_2 + \text{tr} P_-\rho_1(1 - \rho_2). \quad (23)$$

Now we notice that, if two non-negative numbers are greater than the third one, then so is the geometric mean of the first two numbers. Using this fact we combine (22) and (23) to get

$$\begin{aligned} \sqrt{\text{tr} \rho_1(1 - \rho_1)} \sqrt{\text{tr} \rho_2(1 - \rho_2)} \geq \text{tr} P_+(1 - \rho_1)\rho_2 \\ + \text{tr} P_-\rho_1(1 - \rho_2). \end{aligned} \quad (24)$$

On the other hand we can rewrite the trace distance with the use of projectors P_+ and P_- :

$$\begin{aligned} D_{\text{tr}}(\rho_1, \rho_2) &= \frac{1}{2}(\text{tr} P_+\rho_1 - \text{tr} P_+\rho_2 + \text{tr} P_-\rho_2 - \text{tr} P_-\rho_1) \\ &= \frac{1}{2}(\text{tr} P_+\rho_1 + \text{tr} P_+\rho_2 + \text{tr} P_-\rho_2 + \text{tr} P_-\rho_1) - \text{tr} P_+\rho_2 \\ &\quad - \text{tr} P_-\rho_1 = 1 - \text{tr} P_+\rho_2 - \text{tr} P_-\rho_1. \end{aligned} \quad (25)$$

Now finally we can write

$$\begin{aligned} \frac{1}{2} \text{tr}|\rho_1 - \rho_2| + \sqrt{1 - \text{tr} \rho_1^2} \sqrt{1 - \text{tr} \rho_2^2} \\ \geq 1 - \text{tr} P_+\rho_2 - \text{tr} P_-\rho_1 + \text{tr} P_+(1 - \rho_1)\rho_2 + \text{tr} P_-\rho_1(1 - \rho_2) \\ = 1 - \text{tr} \rho_1 \rho_2, \end{aligned} \quad (26)$$

which is equivalent to (16). ■

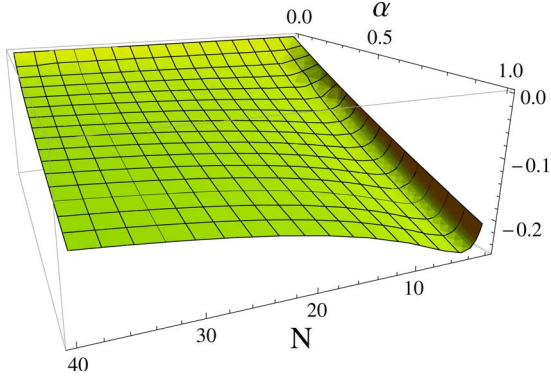


FIG. 1. (Color online) The difference $G(\rho_\alpha, 1/N) - \sqrt{F(\rho_\alpha, 1/N)}$ calculated for states (27) and maximally mixed states as a function of the dimension N and the parameter α [see Eq. (27)].

IV. COMPARISON OF BOUNDS

It is natural to consider the relation between the lower bounds in (3) and (5). It is clear that bound (5) is better than (3) whenever $G - \sqrt{F} \leq 0$.

In the one-qubit case or if one of the states is pure the bound (5) is always better than (3). This follows from the equality between fidelity and superfidelity in these situations [9]. On the other hand the inequality (3) provides a better lower bound if the states ρ_1 and ρ_2 have orthogonal supports. In this case the fidelity between states vanishes, but the superfidelity is not necessarily equal to zero [9].

To get some feeling about the difference between the bound given by fidelity and the present bound we will consider the following families of states.

(i) The family ρ_α is defined as

$$\rho_\alpha = \alpha |\psi\rangle\langle\psi| + (1 - \alpha) \mathbb{1}/N, \quad (27)$$

where $|\psi\rangle\langle\psi|$ is a pure state and in our case we take $|\psi\rangle = |0\rangle \in \mathbb{C}^N$.

(ii) The family $\sigma_\beta \in \Omega_8$ is defined as

$$\sigma_\beta = \beta |\text{GHZ}\rangle\langle\text{GHZ}| + (1 - \beta) \mathbb{1}/N, \quad (28)$$

where $|\text{GHZ}\rangle = (1/\sqrt{2})(|000\rangle + |111\rangle)$.

(iii) The family $\tau_\gamma \in \Omega_8$ is defined as

$$\tau_\gamma = \gamma |010\rangle\langle 010| + (1 - \gamma) \mathbb{1}/N. \quad (29)$$

In Fig. 1 we consider the family ρ_α and calculate the difference $G(\rho_\alpha, 1/N) - \sqrt{F(\rho_\alpha, 1/N)}$. One can see that for small dimensions and states close to the pure state $|\psi\rangle\langle\psi|$ the superfidelity gives a much better approximation for the trace distance than the fidelity. For larger dimensions this is not the case, but nevertheless we can observe that still the superfidelity provides a better bound. This advantage is lost for states close to the maximally mixed state.

A similar situation can be observed in Fig. 2 where the difference between $G(\rho_\alpha, \sigma_\beta)$ and $\sqrt{F(\rho_\alpha, \sigma_\beta)}$ ($\rho_\alpha, \sigma_\beta \in \Omega_8$) is presented. For this particular family the bound (5) is better than (3), but the difference vanishes for states close to the maximally mixed state.

Unfortunately, the bound (5) is not always tighter when compared with (3). Figure 3 shows the difference between

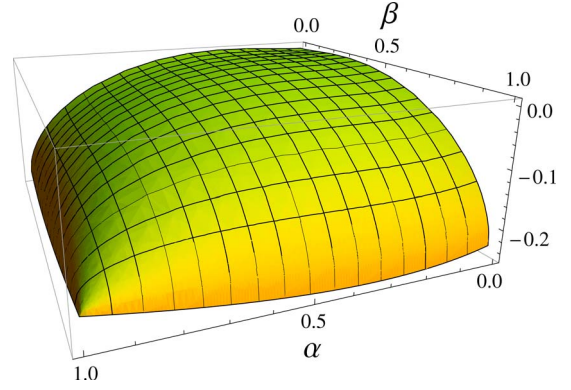


FIG. 2. (Color online) The difference $G(\rho_\alpha, \sigma_\beta) - \sqrt{F(\rho_\alpha, \sigma_\beta)}$ calculated for states (27) and (28) as a function of parameters α and β .

$G(\rho_\alpha, \tau_\gamma)$ and $\sqrt{F(\rho_\alpha, \tau_\gamma)}$ ($\rho_\alpha, \tau_\gamma \in \Omega_8$). As one can see for this family of states there are regions for which the bound (5) is better than (3) but in this case for highly mixed states (3) is better than (5).

V. FINAL REMARKS

We know that the probability of error for distinguishing two density matrices $\rho_1, \rho_2 \in \Omega_N$ is expressed by the trace distance as [1,3]

$$P_E(\rho_1, \rho_2) = \frac{1}{2} [1 - D_{\text{tr}}(\rho_1, \rho_2)]. \quad (30)$$

Using the inequality (5) we can write

$$\frac{1}{2} G(\rho_1, \rho_2) \geq P_E(\rho_1, \rho_2). \quad (31)$$

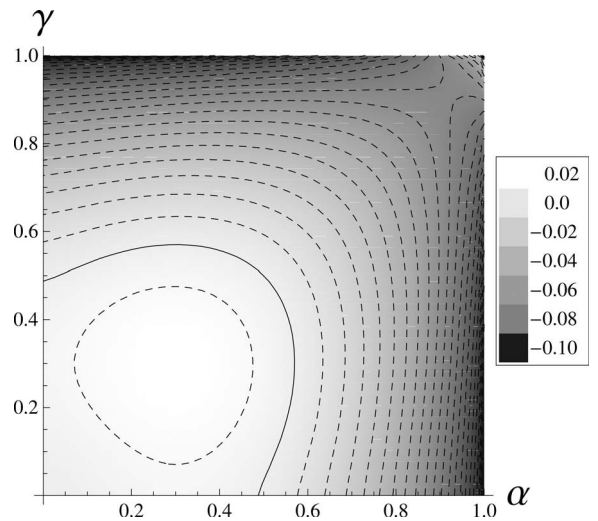


FIG. 3. The difference $G(\rho_\alpha, \tau_\gamma) - \sqrt{F(\rho_\alpha, \tau_\gamma)}$ calculated for states (27) and (29) ($\rho_\alpha, \tau_\gamma \in \Omega_8$) as the function of parameters α and γ . Solid line is a border that separates two regions. For the parameters in the lighter region the inequality (3) provides better bound, while in the darker region the inequality (5) is better.

We have shown the relation between the superfidelity and trace distance, which is analogous to the relation with trace distance and fidelity. This shows that superfidelity can be used to conclude about the distinguishability of states.

Experimental scheme proposed in [9] can be used to estimate superfidelity. Consequently it is possible to estimate experimentally provided lower bound for trace distance.

Our bound provides the relation between the trace distance and the overlap of two operators supplementary to inequality from [[11], Th. 1]. As such it provides a neat mathematical

tool which can be used in quantum information theory.

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