Restricted numerical range: A versatile tool in the theory of quantum information

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Numerical range of a Hermitian operator X is defined as the set of all possible expectation values of this observable among a normalized quantum state. We analyze a modification of this definition in which the expectation value is taken among a certain subset of the set of all quantum states. One considers, for instance, the set of real states, the set of product states, separable states, or the set of maximally entangled states. We show exemplary applications of these algebraic tools in the theory of quantum information: analysis of k-positive maps and entanglement witnesses, as well as study of the minimal output entropy of a quantum channel. Product numerical range of a unitary operator is used to solve the problem of local distinguishability of a family of two unitary gates. © *2010 American Institute of Physics.* [doi:10.1063/1.3496901]

I. INTRODUCTION

Expectation value of a Hermitian observable X among a given pure state $|\psi\rangle$ belongs to the basic notions of quantum theory. It is easy to see that the set Λ of all possible expectation values of a given operator X among all normalized states forms a close interval between the smallest and the largest eigenvalue, $\Lambda(X)=[\lambda_{\min},\lambda_{\max}]$.

In the theory of matrices and operators, one calls such a set *numerical range* or *field of values* of an operator X, which, in general, needs not to be Hermitian.[1,2] Properties of numerical range are intensively studied in the mathematical literature,[3,4] several generalizations of this notion were investigated,[5–8] and its usefulness in quantum theory has been emphasized.[9]

Let us introduce the set Ω of all density matrices of size N, which are Hermitian, positive, and normalized, $\Omega := \{\rho : \rho^\dagger = \rho \geq 0,\ \mathrm{Tr}\,\rho = 1\}$. If a given state is pure, $\rho = |\psi\rangle\langle\psi|$, the expectation value reads $\mathrm{Tr}\,\rho X = \langle\psi|X|\psi\rangle$. Any density matrix can be represented as a convex combination of pure states. Hence, for any operator, the sets of its expectation values among pure states and among mixed states are equal.

More formally, let X be an arbitrary operator acting on an N-dimensional Hilbert space $\mathcal{H}_N$. Its *numerical range* can be defined as

$$\Lambda(X) = \{\mathrm{Tr}\, X\rho, \rho \in \Omega\}. \tag{1}$$

The related concept of *numerical radius*,

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TABLE I. Examples of *restricted* numerical range (NR): $\Lambda_R(X)=\{\mathrm{Tr}\, X\rho, \rho\in\Omega_R\}$, where $\Omega_R\subset\Omega$ denotes a subset of the set of all quantum states of size N. All pure states are assumed to be normalized, $\langle\psi|\psi\rangle=1$, while all coefficients in the sums are non-negative. In each case Eq. (4) provides example of the corresponding notion of restricted numerical radius r_R.

Restricted NR	$\Omega_R\subset\Omega:=\{\rho:\rho^\dagger=\rho\geq 0, \mathrm{Tr}\,\rho=1\}$	Dimension N
NR restricted to real states	$\Omega_R=\{\|\psi\rangle\langle\psi\|, \|\psi\rangle\in\mathbb{R}^N\}$	Arbitrary
Product NR	$\Omega_R=\{\|\psi\rangle\langle\psi\|, \|\psi\rangle=\|\phi^A\otimes\phi^B\rangle\}$	$K\times M$
Separable NR	$\Omega_R=\{\Sigma_i p_i\|\psi_i\rangle\langle\psi_i\|, \|\psi_i\rangle=\|\phi_i^A\otimes\phi_i^B\rangle\}$	$K\times M$
Schmidt rank k NR	$\Omega_R=\{\Sigma_i p_i\|\psi_i\rangle\langle\psi_i\|, \|\psi_i\rangle=\Sigma_{j=1}^k q_{ij}\|\phi_{ij}^A\otimes\phi_{ij}^B\rangle\}$	$K\times M$
Liouville NR	$\Omega_R=\{\|\psi\rangle\langle\psi\|, \|\psi\rangle=\Sigma_{ij}\sigma_{ij}\|i,j\rangle, \sigma^\dagger=\sigma\geq 0, \mathrm{Tr}\,\sigma=1\}$	$M\times M$
$SU(K)$ coherent states NR	$\Omega_R=\{\|\psi\rangle\langle\psi\|, \|\psi\rangle\in SU(K)\text{coherent states}\}$	$(K+l-1)!/l!(K-1)!,\ l\in\mathbb{N}$

$$r(X) = \{|z|, z \in \Lambda(X)\}, \tag{2}$$

is also a frequent subject of study[3,4] (cf. Table II in Sec. V).

In this paper we analyze a modification of standard definitions (1) and (2). For any operator X, one defines its *restricted numerical range*,

$$\Lambda_R(X) = \{\mathrm{Tr}\, X\rho, \rho \in \Omega_R \subset \Omega\}, \tag{3}$$

and the *restricted numerical radius*,

$$r_R(X) = \{|z|, z \in \Lambda_R(X)\}. \tag{4}$$

The symbol Ω_R denotes an arbitrary subset of the set Ω of all normalized density matrices of size N. Thus, the above definition of the restricted numerical range is more general than the one studied in Refs. 2 and 10, in which a subset of the set of pure states was used.

Some examples of restricted numerical ranges are listed in Table I. The range restricted to real states was recently discussed by Holbrook,[11] while the Liouville numerical range, in which the pure states of size M^2 reshaped into a square matrix form a legitimate density operator, was analyzed by Silva.[12] The numerical range of a density matrix ρ restricted to the $SU(2)$ coherent states gives the set of values taken by its Husimi representation—see, e.g., Ref. 13. Examples of restricted numerical radii can be found in Table II at the end of the paper. An very important example is the product numerical radius $r^{\otimes}(X)=\max\{|\langle\psi_1\otimes\ldots\otimes\psi_m|X|\psi_1\otimes\ldots\otimes\psi_m\rangle| : |\psi_i\rangle\in\mathcal{H}_{n_i}\}$, which coincides for $m=2$ and X normal with the Schmidt operator norm $\|X\|_{S(1)}$ introduced by Johnston and Kribs.[14]

TABLE II. Standard algebraic definition of the numerical range and related concepts compared with their *product analogs*. The definitions on the left concern an operator X acting on Hilbert space $\mathcal{H}_N$, while their product analogs are defined for operators acting on a tensor product Hilbert space $\mathcal{H}_{n_1}\otimes\ldots\otimes\mathcal{H}_{n_m}$. Here $|\varphi\rangle$ denotes an arbitrary state of $\mathcal{H}_N$, while $|\psi_1\otimes\ldots\otimes\psi_m\rangle=|\psi_1\rangle\otimes\ldots\otimes|\psi_m\rangle\in\mathcal{H}_1\otimes\ldots\mathcal{H}_m$ represents an arbitrary product state of the composite, m-particle system.

Standard definitions (for simple systems) $X:\mathcal{H}_N\to\mathcal{H}_N$	Product definitions (for multipartite systems) $X:\mathcal{H}_{n_1}\otimes\ldots\otimes\mathcal{H}_{n_m}\to\mathcal{H}_{n_1}\otimes\ldots\otimes\mathcal{H}_{n_m}$
Numerical range $\Lambda(X)=\{\langle\varphi\|X\|\varphi\rangle : \|\varphi\rangle\in\mathcal{H}\}$	Product numerical range $\Lambda^{\otimes}(X)=\{\langle\psi_1\otimes\ldots\otimes\psi_m\|X\|\psi_1\otimes\ldots\otimes\psi_m\rangle : \|\psi_i\rangle\in\mathcal{H}_{n_i}\}$
Numerical radius $r(X)=\max\{\|z\| : z\in\Lambda(X)\}$	Product numerical radius $r^{\otimes}(X)=\max\{\|z\| : z\in\Lambda^{\otimes}(X)\}$
C-numerical range $\Lambda^C(X)=\{\lambda:\lambda=\mathrm{tr}UXU^\dagger C\}$ where $U\in U(N)$	Product C-numerical range $\Lambda_C^{\otimes}(X)=\{\mathrm{tr}(U_1\otimes\ldots\otimes U_k)X(U_1\otimes\ldots\otimes U_k)^\dagger C\}$ where $U_i\in U(n_i)$.
C-numerical radius $r_C(X)=\max\{\|z\| : z\in\Lambda^C(X)\}$	Product C-numerical radius $r_C^{\otimes}(X)=\max\{\|z\| : z\in\Lambda_C^{\otimes}(X)\}$
Higher rank numerical range $\Lambda_l(X)=\{\lambda: P_lXP_l=\lambda P_l\}$ $P_l=\Sigma_{i=1}^l\|i\rangle\langle i\|$	Higher rank product numerical range $\Lambda_l^{\otimes}(X)=\{\lambda: P_l^{\otimes}XP_l^{\otimes}=\lambda P_l^{\otimes}\}$ $P_l^{\otimes}=\Sigma_{i=1}^l\|\underbrace{i\otimes\ldots\otimes i}_{m}\rangle\langle\underbrace{i\otimes\ldots\otimes i}_{m}\|$

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If the dimension of the Hilbert space is a composite number, $N=KM$, the space can be endowed with a tensor product structure,

$$\mathcal{H}_N = \mathcal{H}_K \otimes \mathcal{H}_M. \tag{5}$$

From a physical perspective this corresponds to distinguishing two subsystems in the entire system. One defines then the set of separable pure states, i.e., the states with the product structure, $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$.

Substituting this set into definition (3) of the restricted product range, one arrives at the notion of *product numerical range* of an operator X,

$$\Lambda^{\otimes}(X) = \{\langle \psi_A \otimes \psi_B | X | \psi_A \otimes \psi_B \rangle : |\psi_A\rangle \in \mathcal{H}_K, |\psi_B\rangle \in \mathcal{H}_M\}, \tag{6}$$

where both states $|\psi_A\rangle \in \mathcal{H}_K$ and $|\psi_B\rangle \in \mathcal{H}_M$ are normalized.

The product numerical range can also be considered as a particular case of the decomposable numerical range[5,6] defined for operators acting on a tensor product Hilbert space. This notion was recently analyzed in Refs. 15–18, where the name *local numerical range* was used. In physics context the word "local" refers to local action, so the unitary matrix with a tensor product structure, $U(M) \otimes U(K)$, is said to act "locally" on both subsystems. To be consistent with the mathematical terminology we will use here the name "product numerical range," although a longer version "local product numerical range" would be even more accurate. Note that one may also use other restricted sets of quantum states as these were mentioned in Table I.

The main aim of this work is to demonstrate usefulness of the restricted numerical range for various problems of the theory of quantum information. This paper is organized as follows. In Sec. II we review some basic features of product numerical range and present some examples obtained for Hermitian and non-Hermitian operators. Although we mostly discuss the simplest case of a twofold tensor product structure, which describes the physical case of a bipartite system, we analyze also operators representing the multipartite systems. In Sec. III we study the notion of *separable numerical range* and other *restricted* numerical ranges of an operator acting on a composed Hilbert space.

Key results of this work are presented in Sec. IV in which some applications in the theory of quantum information are presented. In particular, by analyzing a family of one-qubit maps, we find the conditions under which the map is positive and establish a link between product numerical range of a Hermitian operator and the minimum output entropy of a quantum channel. The problem of k-positivity of a quantum map is shown to be connected with properties of the numerical range of the corresponding Choi matrix restricted to the set $\Omega^{(\mathrm{k})}$ of states with the Schmidt number not larger than k.[19] For $k=2$, we point out that the question of distillability of an entangled quantum state is related to the numerical range restricted to the set $\Omega^{(2)}$.

Furthermore, properties of product numerical range of non-Hermitian operators are used to solve the problem of local distinguishability for a family of two-qubit gates. In Sec. V we present some concluding remarks and discuss further possibilities of generalizations of numerical range which could be useful in quantum theory. Proofs of certain lemmas are relegated to Appendix.

II. PRODUCT NUMERICAL RANGE

Quantum information theory deals with composite quantum systems which can be described in a complex Hilbert space with a tensor product structure.[20] When analyzing properties of operators acting on composed Hilbert space (5), it is physically justified to distinguish *product* properties, which reflect the structure of the Hilbert space.

If the physical system is isolated from the environment, its dynamics in time can be described by a unitary evolution $|\psi'\rangle = U|\psi\rangle$, where U is unitary, $UU^{\dagger} = \mathbb{1}_N$. In the case of a bipartite system, $N=KM$, one distinguishes a class of *local dynamics*, which take place independently in both physical subsystems, so that $U=U_A \otimes U_B$, where $U_A \in U(K)$, while $U_B \in U(M)$. From a group-theoretical perspective, one distinguishes the direct product $U(K) \times U(M)$, which forms a proper subgroup of $U(KM)$.

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It is important to know which tasks, such as the discrimination of pure quantum states, can be completed with the use of local operations and classical communication. For this purpose, it is convenient to work with the notion of the product numerical range of an operator defined by Eq. (6). This algebraic tool can be considered as a natural generalization of the standard numerical range for operators acting on a tensor product Hilbert space.

Note that the definition of *product numerical range* is not unitarily invariant, but implicitly depends on the particular decomposition of the Hilbert space. This notion may also be considered as a numerical range *relative* to the proper subgroup $U(K)\times U(M)$ of the full unitary group $U(KM)$. It is worth mentioning that product numerical range differs from the so-called *quadratic numerical range*, also defined for operators acting on a composite Hilbert space.[21]

Consider the following problems arising in the theory of quantum information.

- Verify if a given map Φ acting on the set of quantum states is positive: Is $\Phi(\rho)\geq 0$ for all $\rho\geq 0$?
- For a given observable X, defined for a bipartite system, find the largest (the smallest) expectation value among pure product states: What is $\max_{|\phi_A,\phi_B\rangle}\langle\phi_A,\phi_B|X|\phi_A,\phi_B\rangle$?
- Check if two unitary gates U_1 and U_2 acting on a bipartite systems are distinguishable. This is the case if there exists a product state $|\psi\rangle=|\phi_A,\phi_B\rangle$, such that the states $U_1|\psi\rangle$ and $U_2|\psi\rangle$ are orthogonal.
- For a pair of two bipartite states σ and ρ maximize their fidelity or the trace Tr $\rho\sigma$ by the means of local operations.

This list of questions, of different difficulty levels, could be easily extended. All these problems have one thing in common: they could be directly solved, if we had an efficient algorithm to compute the product numerical range of an operator. Although in this work we are not in a position to go so far, we aim to show usefulness of this notion and present some partial results.

A. Basic properties

In this section we review some basic properties of product numerical range. Some of them were discussed by Dirr *et al.*,[15] while some other were established in Ref. 22.

For any operator X acting on a Hilbert space $\mathcal{H}_N$, its product numerical range (6) forms a nonempty, connected set in the complex plane. However, this set needs not to be convex nor simply connected. Further properties of product numerical range include

(a) subadditivity, $\Lambda^{\otimes}(A+B)\subset\Lambda^{\otimes}(A)+\Lambda^{\otimes}(B)$,
(b) translation: for any $\alpha\in\mathbb{C}$ one has $\Lambda^{\otimes}(A+\alpha\mathbb{1})=\Lambda^{\otimes}(A)+\alpha$,
(c) scalar multiplication: for any $\alpha\in\mathbb{C}$ one has $\Lambda^{\otimes}(\alpha A)=\alpha\Lambda^{\otimes}(A)$,
(d) product unitary invariance: $\Lambda^{\otimes}((U\otimes V)A(U\otimes V)^{\dagger})=\Lambda^{\otimes}(A)$,
(e) if A is normal, then numerical range of its tensor product with an arbitrary operator B coincides with the convex hull of the product numerical range, $\Lambda(A\otimes B)=\mathrm{Co}(\Lambda^{\otimes}(A\otimes B))$,
(f) product numerical range of any A contains the *barycenter* of the spectrum, $\frac{1}{KM}\mathrm{tr}\,A\in\Lambda^{\otimes}(A)$.

To analyze product numerical range of the Kronecker product, it is convenient to make use of the geometric algebra of complex sets.[23] For any two sets Z_1 and Z_2 on the complex plane, one defines their *Minkowski product*,

$$Z_1\boxtimes Z_2=\{z:z=z_1z_2,z_1\in Z_1,z_2\in Z_2\}. \tag{7}$$

Observe that this operation is not denoted by the standard symbol $\otimes$ in order to avoid the risk of confusion with the tensor product of operators. The above definition allows us to express the product numerical range of the Kronecker product of arbitrary two operators as a Minkowski product of the numerical ranges of both factors,[2]

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$$\Lambda^{\otimes}(A \otimes B) = \Lambda(A) \boxtimes \Lambda(B). \tag{8}$$

This property can be directly generalized to an arbitrary number of factors. Thus, the problem of finding the product numerical range of a tensor product can be reduced to finding the Minkowski product[23,24] of two or more numerical ranges.

B. Hermitian case

In the case of a Hermitian operator $X=X^{\dagger}$ acting on $\mathcal{H}_N$, its spectrum belongs to the real axis. Labeling the eigenvalues in a weakly increasing order, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$, one can write the numerical range as an interval, $\Lambda(X)=[\lambda_1,\lambda_N]$, see, e.g., Ref. 1.

Let us assume that the Hilbert space has a product structure, $\mathcal{H}_N=\mathcal{H}_K\otimes\mathcal{H}_M$, which implies a notion of a pure product state. Define the points $\lambda^{\otimes}_{\min}$ and $\lambda^{\otimes}_{\max}$ as the maximal and the minimal expectation values of X among all product pure states. Then the product numerical range is given by a closed interval, $\Lambda^{\otimes}(X)=[\lambda^{\otimes}_{\min},\lambda^{\otimes}_{\max}]$. If the spectrum of X is not degenerated to a single point (which is the case if and only if X is proportional to identity), then $\lambda^{\otimes}_{\min}\neq\lambda^{\otimes}_{\max}$, so the product numerical range has a nonzero volume.[22]

Making use of the lemma about the dimensionality of subspaces belonging to a composed Hilbert space of size $N=KM$ which contain at least one separable state,[25] one can get the following bounds for the edges of the product numerical range:

$$\lambda^{\otimes}_{\min} \leq \lambda_{(K-1)(M-1)+1} \quad \text{and} \quad \lambda^{\otimes}_{\max} \geq \lambda_{K+M-1}. \tag{9}$$

These bounds, proven in Ref. 22, imply that in the simplest case of a 2×2 system ($N=4$), the product numerical range contains the central segment of the spectrum,

$$\Lambda(X) = [\lambda_1,\lambda_4] \supset \Lambda^{\otimes}(X) = [\lambda^{\otimes}_{\min},\lambda^{\otimes}_{\max}] \supset [\lambda_2,\lambda_3]. \tag{10}$$

Similarly, for any Hermitian X acting on a $2\times K$ space, the central segment of the spectrum $[\lambda_K,\lambda_{K+1}]$ belongs to $\Lambda^{\otimes}(X)$. In the case of a 3×3 system ($N=9$), the product numerical range of X contains its central eigenvalue, $\lambda_5 \in \Lambda^{\otimes}(X)$.

1. Exemplary Hermitian matrix of order four

Not being able to construct an algorithm to obtain product numerical range for an arbitrary Hermitian operator, we shall study some concrete examples. Consider first positive numbers $t,s \geq 0$ and a family of Hermitian matrices of order of 4,

$$X_{t,s} = \begin{pmatrix} 2 & 0 & 0 & t \\ 0 & 1 & s & 0 \\ 0 & s & -1 & 0 \\ t & 0 & 0 & -2 \end{pmatrix}, \tag{11}$$

with the spectrum

$$\{-\sqrt{s^2+1}, \sqrt{s^2+1}, -\sqrt{t^2+4}, \sqrt{t^2+4}\}. \tag{12}$$

Then we can write

$$\langle x| \otimes \langle y|X_{t,s}|x\rangle \otimes |y\rangle = 2|x_1|^2|y_1|^2 + |x_1|^2|y_2|^2 - |x_2|^2|y_1|^2 - 2|x_2|^2|y_2|^2 + 2tRe[x_2^* x_1 y_2^* y_1] + 2sRe[x_1^* x_2 y_1^* y_2]. \tag{13}$$

Because in the case of Hermitian matrices, the product numerical range forms a closed interval, we only need to find the upper and the lower bounds for the above expression. We have

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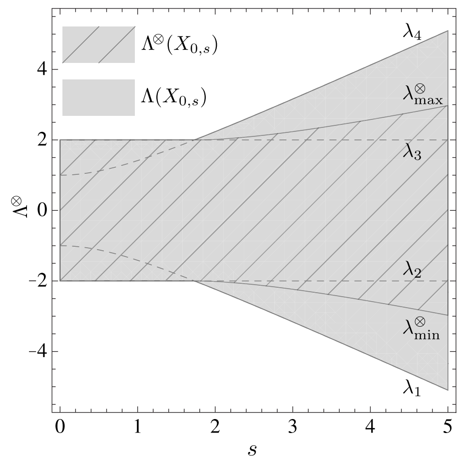

FIG. 1. Numerical range and product numerical range for matrices $X_{0,s}$, which belong to family (11).

$$\langle x| \otimes \langle y|X_{t,s}|x\rangle \otimes |y\rangle \leq 2|x_1|^2|y_1|^2 + |x_1|^2|y_2|^2 - |x_2|^2|y_1|^2 - 2|x_2|^2|y_2|^2 + 2(t+s)|x_2||x_1||y_2||y_1|. \tag{14}$$

Because $|x_1|^2+|x_2|^2=1$, we can put $p=|x_1|^2$ and $q=|y_1|^2$ with $p,q\geq 0$. This gives us

$$\langle x| \otimes \langle y|X_{t,s}|x\rangle \otimes |y\rangle \leq 2pq + p(1-q) - (1-p)q - 2(1-p)(1-q) + 2(t+s)\sqrt{p(1-p)q(1-q)}. \tag{15}$$

We want to maximize the above expression under the following constraints: $0\leq p\leq 1$ and $0\leq q \leq 1$.

First we analyze the edge. On the edge (one of the variables p,q is 0 or 1) the square root vanishes, the remaining part is convex and thus the extreme points are $(p,q) \in \{(0,0),(0,1),(1,0),(1,1)\}$. Thus, the maximum value on the edge is 2. If we assume that $\underline{p,q\notin\{0,1\}},$ we have to find zeros of appropriate derivatives. The extremum value is $\sqrt{t^4+10t^2+9}/2t$ for $t\geq\sqrt{3}$. The lower estimate is obtained similarly. Thus the exact formula for the product numerical range reads

$$\Lambda^{\otimes}(X_{t,s}) = [-f(t+s), f(t+s)], \tag{16}$$

where

$$f(t) = \begin{cases} 2 & \text{for } t \in [0,\sqrt{3}) \\ \sqrt{t^4+10t^2+9}/2t & \text{for } t \in [\sqrt{3},\infty). \end{cases} \tag{17}$$

Note that the product numerical range depends only on the sum of the parameters s and t, whereas the numerical range depends on the values of both of them. The minimum and the maximum values in the numerical range and the product numerical range of the matrix $X_{t,s}$ are compared in Fig. 1.

Let us consider a more general family of matrices for $t,s\geq 0$,

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$$Y_{t,s} = \begin{pmatrix} a & 0 & 0 & t \\ 0 & b & s & 0 \\ 0 & s & c & 0 \\ t & 0 & 0 & d \end{pmatrix}. \tag{18}$$

For given $a, \ b, \ c, \ d$, one can obtain a similar result as above, but, in general, the formulas are very complex due to the higher number of parameters. However, it is easy to obtain the following bound:

$$[f(s+t), g(s+t)] \subset \Lambda^{\otimes}(Y_{s,t}), \tag{19}$$

where

$$f(t) = \min\left\{ \min(a,b,c,d), \frac{1}{4}\mathrm{tr}\ Y_{0,0} - \frac{1}{2}t \right\} \tag{20}$$

and

$$g(t) = \max\left\{ \max(a,b,c,d), \frac{1}{4}\mathrm{tr}\ Y_{0,0} + \frac{1}{2}t \right\}. \tag{21}$$

2. A tridiagonal Hermitian matrix

Consider another family of Hermitian matrices of size 4, written in the standard product basis,

$$D = \begin{bmatrix} \frac{1}{2} & a & 0 & 0 \\ a^* & \frac{1}{2} & b & 0 \\ 0 & b^* & \frac{1}{2} & c \\ 0 & 0 & c^* & \frac{1}{2} \end{bmatrix}, \tag{22}$$

where a and b are arbitrary complex numbers and $c = xa$ for some arbitrary real number x.

This family was introduced in Ref. 26 as a useful example for studying block-positivity. Here we deal with the product numerical range of D, but the two concepts are closely related, since a Hermitian matrix acting on a bipartite Hilbert space is block-positive if and only if its product numerical range belongs to $\mathbb{R}_+$. Following the lines of Ref. 26, with some additional effort, one obtains an explicit result,

$$\Lambda^{\otimes}(D) = \left[\frac{1}{2} - G, \frac{1}{2} + G \right], \tag{23}$$

where

$$G = \frac{1}{4}(|a+c| + \sqrt{|a-c|^2 + |b|^2}). \tag{24}$$

3. Family of isospectral Hermitian operators

It is instructive to study product numerical range for a family of Hermitian operators with a fixed spectrum and varying eigenvectors. Any unitary 4×4 matrix U may be represented in a canonical form,

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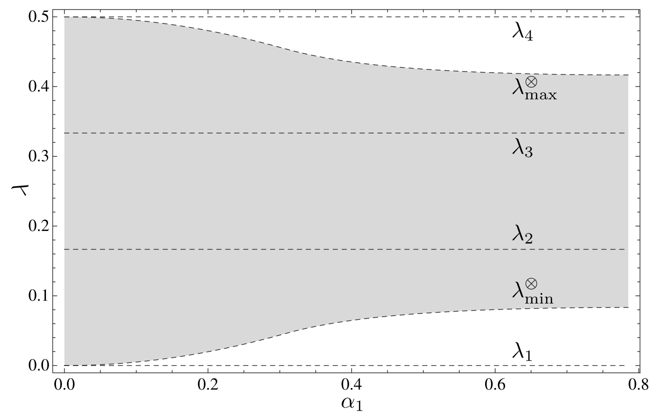

FIG. 2. Product eigenvalues and product numerical range (gray region) of the one-parameter (α_1) family of matrices given by Eq. (27) with eigenvalues $\lambda_1=0$, $\lambda_2=1/6$, $\lambda_3=1/3$, $\lambda_4=1/2$.

$$U=(V_A\otimes V_B)U_d(W_A\otimes W_B), \tag{25}$$

where $V_A,V_B,W_A,W_B\in U(2)$, while U_d is a unitary matrix of size four expressed in the form[27]

$$U_d(\alpha_1,\alpha_2,\alpha_3)=\exp\left(i\sum_{k=1}^{3}\alpha_k\sigma_k\otimes\sigma_k\right). \tag{26}$$

Here σ_k denotes the Pauli matrices, and the three real parameters α_i belong to the interval $[0,\frac{\pi}{4}]$.

Consider a density matrix obtained from the diagonal matrix $E(x_1,x_2,x_3)=\mathrm{diag}(x_1,x_2,x_3,1-x_1-x_2-x_3)$ by a nonlocal unitary rotation,

$$\rho(\alpha_1,x_i)=U_dEU_d^{\dagger}, \tag{27}$$

with $\alpha_2=\alpha_3=0$. Figure 2 presents the dependence of its product numerical range as a function of the nonlocality phase α_1.

4. Random Hermitian matrices of order four

As shown in the above examples, the lower edge of the product numerical range of a Hermitian matrix X of order four is interlaced between its two smallest eigenvalues, $\lambda^{\otimes}_{\min}\in[\lambda_1,\lambda_2]$. We have already seen that these bounds can be saturated, so the exact position of $\lambda^{\otimes}_{\min}$ is X dependent. However, following the statistical approach, one may pose the question how the edge is located with respect to both eigenvalues for a *random* Hermitian operator.

To analyze this problem we generated numerically a 5×10^5 random Hermitian matrices according to the flat (Hilbert–Schmidt) measure in the set Ω of normalized density matrices of size $N=4$. The joint probability distribution for the eigenvalues reads[28]

$$P(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\frac{15!}{3456}\delta\left(1-\sum_{j=1}^{4}\lambda_j\right)\prod_{i<j}^{4}(\lambda_i-\lambda_j)^2. \tag{28}$$

By construction, the eigenvalues sum to unity, and this normalization sets the scale. It is possible to integrate out of the above formula any chosen three eigenvalues and obtain an explicit probability distribution for the last one. For instance, the distribution for the smallest eigenvalue has the form

$$P(\lambda_1)=60(1-4\lambda_1)^{14}\Theta(\lambda_1)\Theta(1/4-\lambda_1), \tag{29}$$

where $\Theta(x)$ is the Heaviside function.

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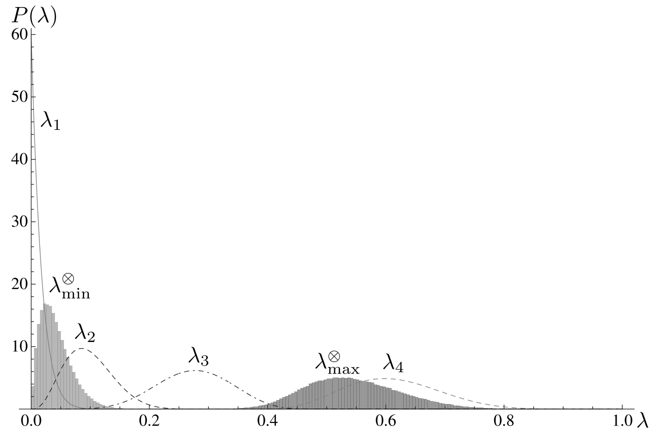

FIG. 3. Probability density of eigenvalues λ_1, λ_2, λ_3, λ_4 (dashed/dotted lines) and product values $\lambda^{\otimes}_{\rm min}$, $\lambda^{\otimes}_{\rm max}$ (dark histograms) for a random two-qubit density matrix, generated according to Hilbert–Schmidt measure (28).

Figure 3 presents the probability distributions for ordered eigenvalues, $\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4$, obtained analytically by integration of (28). These distributions are compared with the distributions $P(\lambda^{\otimes}_{\rm min})$ and $P(\lambda^{\otimes}_{\rm max})$ obtained numerically. As follows from (10), $\lambda^{\otimes}_{\rm min}$ is located between the two smallest eigenvalues, while $\lambda^{\otimes}_{\rm max}$ is interlaced by the two largest eigenvalues λ_3 and λ_4. Note that the histogram is not symmetric with respect to the change $\lambda_1 \leftrightarrow \lambda_4$ and $\lambda^{\otimes}_{\rm min} \leftrightarrow \lambda^{\otimes}_{\rm max}$, since the eigenvalues are ordered, so the mean distance of the smallest eigenvalue to zero is smaller than the mean distance of the largest eigenvalue to unity.

C. Non-Hermitian case and Multipartite operators

The above analysis can be extended in a natural way for Hilbert spaces with m-fold tensor product structure, used to describe quantum systems consisting of m subsystems,

$$\mathcal{H}_N = \mathcal{H}_{n_1} \otimes \cdots \otimes \mathcal{H}_{n_m}, \tag{30}$$

with $N=n_1 \ldots n_m$. In the case of an operator X acting on this space, its product numerical range consists of all expectation values $\langle \psi_{\rm prod}|X|\psi_{\rm prod}\rangle$ among pure product states, $|\psi_{\rm prod}\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_m\rangle$.

If the number m of subsystems is larger than 2, there exist operators for which product numerical range forms a set which is not simply connected.[17,22] In fact, the genus of this set can be greater than 1. To show an illustrative example, we consider a unitary matrix of size 2,

$$U = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}. \tag{31}$$

The product numerical range of $U^{\otimes n}$ can be found analytically for any integer n by applying an extension of formula (8) to multipartite systems. Numerical range of U forms an interval I joining the complex eigenvalue $e^{i\phi}$ with the unity. Thus, to find $\Lambda^{\otimes}(U^{\otimes n})$, it suffices to compute the n-fold Minkowski power of the interval I on the complex plane. More explicitly, $\Lambda^{\otimes}(U^{\otimes n})$ consists of all the points $z_1 z_2 \ldots z_n$, where $z_i = 1-\lambda_i+\lambda_i e^{i\phi}$ and $\lambda_i \in [0,1]$ for all $i \in \{1,2,\ldots,n\}$. Let us denote by $f(\alpha)$ the modulus $|z|$ of $z=1-\lambda+\lambda e^{i\phi}$ as a function of the phase $\alpha := \mathrm{Arg}(z)$. Obviously, f is a convex function of α. One can relatively easy get an explicit expression for f,

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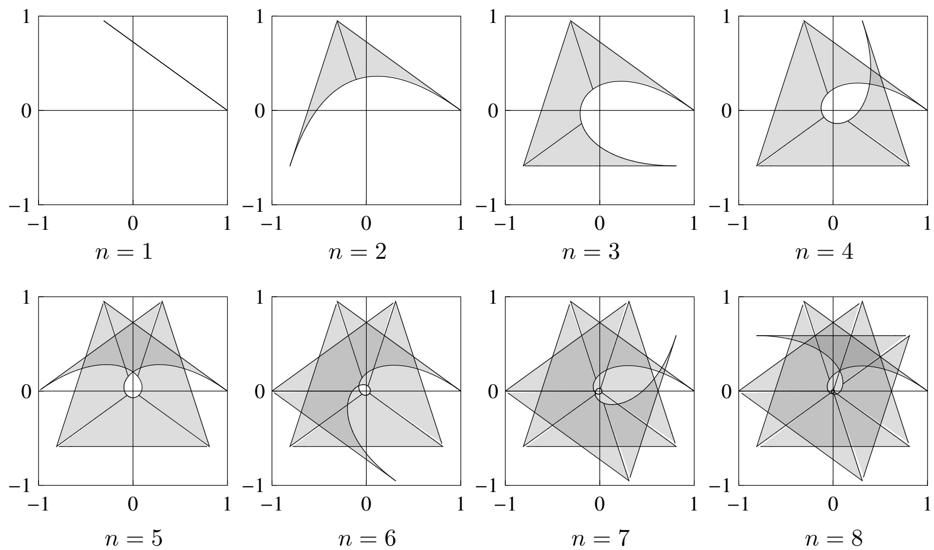

FIG. 4. Product numerical range of $U^{\otimes n}$ for U specified in (31) with $\phi=3\pi/5$ and $n=1,\ldots,8$.

$$f(\alpha)=\frac{\cos(\phi/2)}{\cos(\alpha-\phi/2)}. \tag{32}$$

Thus, the numbers z of the form $z=1-\lambda+\lambda e^{i\phi}$, $\lambda\in[0,1]$ have a parametrization $\alpha\mapsto e^{i\alpha}f(\alpha)$ with $\alpha\in[0,\phi]$ and f given by formula (32). Because of the convexity of f, for a fixed $\mathrm{Arg}(z_1z_2\ldots z_n)$, the minimum of $|z_1z_2\ldots z_n|$ is attained when $z_1=z_2=\ldots=z_n$. The resulting curve marks the border of $\Lambda^{\otimes}(U^{\otimes n})$ and has a parametrization

$$[0,\phi]\ni\alpha\mapsto e^{in\alpha}\left(\frac{\cos(\phi/2)}{\cos(\alpha-\phi/2)}\right)^n. \tag{33}$$

The remaining parts of the border of $\Lambda^{\otimes}(U^{\otimes n})$ are included in the n segments $\{[e^{i(k-1)\phi},e^{ik\phi}]\}_{k=1}^n$. This follows because the maximum of $|z_1z_2\ldots z_n|$ for a fixed $\beta=\mathrm{Arg}(z_1z_2\ldots z_n)$ is attained for $z_1=z_2=\ldots=z_{k-1}=e^{i\phi}$, $z_k=e^{i(\beta-k\phi)}$, and $z_{k+1}=z_{k+2}=\ldots=z_n=1$, where $k=\lfloor\beta/\phi\rfloor$. In Fig. 4 we choose $\phi=\frac{3\pi}{5}$ and plot the product numerical ranges of $U^{\otimes n}$ for $n=1,2,\ldots,8$.

Observe that if 0 does not belong to the numerical range of U, it does not belong to the product numerical range of $U^{\otimes n}$. Hence if for sufficiently large exponent n the product range "wraps around" zero, the set $\Lambda^{\otimes}(U^{\otimes n})$ is not simply connected.

As one may notice in Fig. 4, it is possible to construct a tensor product of operators, such that its product numerical range has genus 2. If we magnify picture number 7 from Fig. 4, it becomes evident that the genus of $\Lambda^{\otimes}(U^{\otimes 7})$ is equal to 2 (cf. Fig. 5). Observe that if n is further increased, the genus of $\Lambda^{\otimes}(U^{\otimes n})$ is not smaller than 1, although the size of the hole around $z=0$ shrinks exponentially fast. More precisely, the distance between the set $\Lambda(U)$ and zero is $\cos(\phi/2)$, which implies that the distance between $\Lambda^{\otimes}(U^{\otimes n})$ and zero equals $[\cos(\phi/2)]^n$ for arbitrary n.

In general, finding the product numerical range of a non-Hermitian operator without the tensor product structure is not a simple task. However, in the special case of a normal operator X, which can be diagonalized by product of unitary matrices, a useful parametrization of its product numerical range was described in Ref. 22.

III. SEPARABLE NUMERICAL RANGE

Consider a tensor product Hilbert space $\mathcal{H}_N=\mathcal{H}_K\otimes\mathcal{H}_M$ and the set Ω of all normalized states acting on it, $\rho\in\Omega\Leftrightarrow\rho=\rho^{\dagger},\ \rho\geq 0,\ \mathrm{Tr}\,\rho=1$. One distinguishes its subset Ω_{sep} of *separable states*, i.e., states that can be represented as a convex combination of product states,

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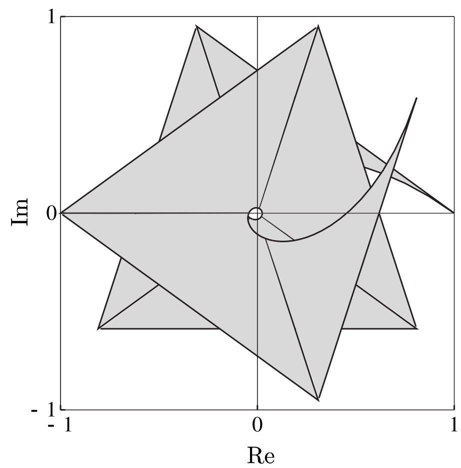

FIG. 5. Product numerical range of $U^{\otimes 7}$ (plotted in gray) forms a set of genus 2. The matrix U is given by Eq. (31) with $\phi=3\pi/5$.

$$\rho \in \Omega_{\text{sep}} \Leftrightarrow \rho \in \Omega \quad \text{and} \quad \rho = \sum_i p_i \rho_i^{(K)} \otimes \rho_i^{(M)}. \tag{34}$$

Here positive coefficients p_i form a probability vector, while $\rho_i^{(K)}$ and $\rho_i^{(M)}$ denote arbitrary states acting on $\mathcal{H}_K$ and $\mathcal{H}_M$, respectively. Any state ρ which cannot be represented in the above form is called *entangled*.[13] Hence this definition depends on the particular choice of the tensor product structure, $\mathcal{H}_N=\mathcal{H}_K\otimes\mathcal{H}_M$.

Observe that Definition 6 of the product numerical range of an operator X acting on $\mathcal{H}_K \otimes \mathcal{H}_M$ can be formulated as

$$\Lambda^{\otimes}(X) = \{\text{Tr } X\rho : \rho = |\psi_A\rangle \otimes |\psi_B\rangle\langle\psi_A| \otimes \langle\psi_B|\}. \tag{35}$$

It is then natural to introduce an analogous definition of *separable numerical range*,

$$\Lambda^{\text{sep}}(X) \coloneqq \{\text{Tr } X\rho : \rho \in \Omega_{\text{sep}}\}. \tag{36}$$

Since any product state is separable, the product numerical range forms a subset of the separable numerical range, $\Lambda^{\otimes}(X) \subset \Lambda^{\text{sep}}(X)$. By definition, the set Ω_{sep} of separable states is convex. This fact allows us to establish a simple relation between both sets.

Proposition 1: Separable numerical range forms the convex hull of the product numerical range,

$$\Lambda^{\text{sep}}(X) = \text{co}(\Lambda^{\otimes}(X)).$$

Proof: Assume that $\lambda \in \text{co}(\Lambda^{\otimes}(X))$, so it can be represented as a convex combination of points belonging to the product numerical range, $\lambda = \Sigma_i p_i \lambda_i$. Taking the convex combination of the corresponding product states $|\phi_i\rangle = |\psi_i^A\rangle \otimes |\psi_i^B\rangle$, we get a separable mixed state $\rho = \Sigma_i p_i |\phi_i\rangle\langle\phi_i|$, such that $\text{Tr } X\rho = \lambda$. A similar reasoning shows that if $\lambda \notin \text{co}(\Lambda^{\otimes}(X))$, there is no separable state ρ, such that $\text{Tr } X\rho = \lambda$. ■

Following Ref. 22, one can note that if A or B is normal then $\Lambda^{\text{sep}}(A\otimes B) = \Lambda(A \otimes B)$.

Since product numerical range of a Hermitian operator forms an interval, in this case the separable and product numerical ranges do coincide. This is not the case in general. A typical example is shown in Figs. 6(b) and 6(c), in which the separable numerical range forms a proper subset of the standard numerical range and includes the product numerical range as its proper subset.

Consider, for instance, a unitary matrix U of size of 4 with a nondegenerate spectrum. Its numerical range is then formed by a quadrangle inscribed into the unit circle. If all eigenvectors of

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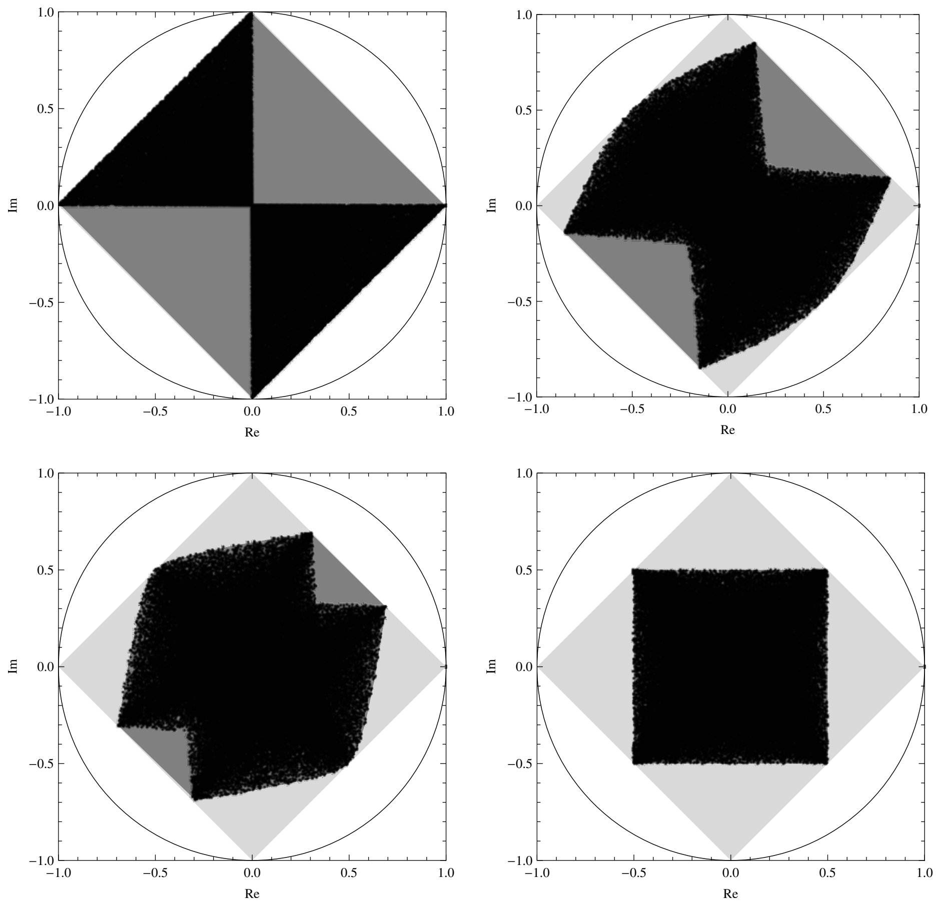

FIG. 6. Numerical range (light gray), separable numerical range (dark gray), and product numerical range (black dots obtained by random sampling) of family of matrices $X_\alpha = U_d(\alpha,0,0)\cdot \mathrm{diag}(i,-1,-i,1)\cdot U_d(\alpha,0,0)^\dagger$, where U_α is given by Eq. (26) for $\alpha=0,\pi/8,3\pi/16,\pi/4$. In the case of $\alpha=0$, the eigenvectors of X form orthonormal canonical basis and X is normal therefore $\Lambda^{\mathrm{sep}}(X)=\Lambda(X)$. In the case of $\alpha=\pi/4$ all eigenvectors of X are maximally entangled states and $\Lambda^{\mathrm{sep}}(X)=\Lambda^{\otimes}(X)$.

this matrix are entangled, the product numerical range of U does not contain any of its eigenvalues. In a generic case $\Lambda^{\otimes}(U)$ is not convex and it forms a proper subset of $\Lambda^{\mathrm{sep}}(U)$—see Fig. 6.

A. *k*–Entangled numerical range

Any pure state in a $N=KM$ dimensional bipartite Hilbert space can be represented by its *Schmidt decomposition*,

$$|\psi\rangle = \sum_{i=1}^{K}\sum_{j=1}^{M} A_{ij}|i\rangle \otimes |j\rangle = \sum_{i=1}^{K} \sqrt{\mu_i}|i'\rangle \otimes |i''\rangle. \tag{37}$$

We the have assumed here that $K\leq M$ and denoted a suitably rotated product basis by $|i'\rangle\otimes|i''\rangle$. The eigenvalues μ_i of a positive matrix $AA^\dagger$ are called the *Schmidt coefficients* of the bipartite state $|\psi\rangle$. The normalization condition $|\psi|^2=\langle\psi|\psi\rangle=1$ implies that $\|A\|_{\mathrm{HS}}^2=\mathrm{tr}\,AA^\dagger=1$, so the Schmidt coefficients μ_i form a probability vector—see, e.g., Ref. 13.

The state $|\psi\rangle$ is separable if and only if the $K\times M$ matrix of coefficients A is of rank 1, so the corresponding vector of the Schmidt coefficients is pure. A given mixed state ρ is called separable if it can be represented as a convex combination of product pure states. This notion can be generalized in a natural way, and in the theory of quantum information,[19] once considers set $\Omega^{(\mathrm{k})}$

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of states which can be decomposed into a convex combination of states with the Schmidt number not larger than k. In symbols, $\rho=\Sigma_{j=1}p_j|\phi_j\rangle\langle\phi_j|$, with all vectors $|\phi_j\rangle=\Sigma_{i=1}^{k}\xi_i|\psi_i^A\rangle\otimes|\psi_i^B\rangle$ of Schmidt rank at most k. We may choose k to be $1,\ldots,K$, where $K=M$ denotes the dimensionality of each subsystem. By definition, $\Omega^{(1)}=\Omega_{\text{sep}}$ represents the set of separable states, while $\Omega^{(\text{K})}=\Omega$ denotes the entire set of mixed quantum states.

Making use of the definition of the subset $\Omega^{(\text{k})}$ of the set of all states in (3), one obtains an entire hierarchy of restricted numerical ranges denoted by $\Lambda^{(k)}$. As the elements of $\Omega^{(\text{k})}$ are called k-entangled states,[29] the set $\Lambda^{(\text{k})}(X)$ will be referred to as numerical range restricted to k-entangled states.

For $k=1$ one has $\Omega^{(1)}=\Omega_{\text{sep}}$ so in this case one obtains the separable numerical range, $\Lambda^{(1)}=\Lambda^{\text{sep}}$. Note that in this convention a 1-entangled state means a separable state. In the other limiting case $k=K$, $\Omega_K=\Omega$ and one arrives at the standard numerical range, $\Lambda^{(K)}=\Lambda$. The following chain of inclusions $\Lambda^{\otimes}\subset\Lambda^{(1)}\subset\Lambda^{(2)}\subset\cdots\subset\Lambda^{(\text{K})}=\Lambda$ holds by construction. This implies inequalities between the corresponding restricted numerical radii, $r^{\otimes}\leq r^{(1)}\leq r^{(2)}\leq\ldots,\leq r^{(K)}=r$.

IV. APPLICATIONS IN QUANTUM INFORMATION THEORY

In this section we link various problems in the theory of quantum information processing which have one thing in common: they can be analyzed using the restricted numerical range or related notions.

A. Block-positive matrices and entanglement witnesses

Let us start by recalling the standard definition of block-positivity.[13] A Hermitian matrix X acting on the tensor product Hilbert space, $\mathcal{H}_N=\mathcal{H}_M\otimes\mathcal{H}_K$, is called *block-positive*, if it is positive on all product states. Making use of the notation introduced in Sec. II, this property reads $\lambda_{\min}^{\otimes}(X)\geq 0$. Therefore, checking if a given Hermitian matrix is block-positive is equivalent to showing that its product numerical range forms a subset of $[0,\infty)$.

Block-positive matrices arise in a characterization of positive quantum maps by the theorem of Jamiołkowski.[30] A map Φ taking operators on $\mathcal{H}_K$ to operators on $\mathcal{H}_M$ is called *positive*, if it maps positive operators to positive operators. Let $|\Psi_+\rangle\langle\Psi_+|$ denote the orthogonal projection onto the maximally entangled state $|\Psi_+\rangle=\frac{1}{\sqrt{K}}\Sigma_{i=1}^{K}|i\rangle|i\rangle$ acting on $\mathcal{H}_K\otimes\mathcal{H}_K$. The Jamiołkowski theorem states that Φ is positive if and only if the corresponding dynamical matrix (Choi matrix[31]), $D_\Phi=(\Phi\otimes\mathbb{1})|\Psi_+\rangle\langle\Psi_+|$, is block-positive. This leads us to the following characterization of positive maps in terms of the product numerical range of D_Φ.

Proposition 2: Let Φ be a linear map taking operators on $\mathcal{H}_K$ to operators on $\mathcal{H}_M$. Then

$$\Phi \ \text{is positive} \Leftrightarrow \Lambda^{\otimes}(D_\Phi)\subset[0,\infty). \tag{38}$$

That is, the product numerical range of D_Φ has to be contained in the positive semiaxis in order for Φ to be positive. As discussed in Sec. III, for any Hermitian D, its product and separable numerical ranges do coincide. Consequently, positivity of Φ can be formulated with $\Lambda^{\text{sep}}(D_\Phi)$. The positivity condition reads $\text{Tr}\, D_\Phi\rho\geq 0$ for any separable ρ. This is the same as $\Lambda^{\text{sep}}(D_\Phi)\subset[0;+\infty)$.

We recall that a map Φ is called k-positive if $\Phi\otimes\mathbb{1}_k$ is a positive map. If this is the case for arbitrary $k\in\mathbb{N}$, the map is called *completely positive*. The famous theorem by Choi[31] concerning completely positive maps can be expressed in a similar manner.

Proposition 3: Let Φ be a linear map taking operators on $\mathcal{H}_K$ to operators on $\mathcal{H}_M$. Then

$$\Phi \ \text{is completely positive} \Leftrightarrow \Lambda(D_\Phi)\subset[0,\infty). \tag{39}$$

The difference is that (38) refers to the product numerical range of D_Φ, whereas (39) concerns the standard numerical range. Note that $\Lambda(D_\Phi)\subset[0,\infty)$ is just another way of writing that D_Φ is a positive operator.

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Positive maps find a direct application in the theory of quantum information due to a theorem by the Horodecki family:[32] a state σ of a bipartite system is separable if and only if $(\Phi \otimes \mathbb{1})\sigma \geq 0$ for any positive map Φ. In the opposite case, the state σ is entangled.

The above results explain recent interest in characterization of the set of positive maps. A block-positive matrix $W := D_\Phi$, which corresponds to a map which is positive but not completely positive, is called an *entanglement witness*, since it can be used to detect quantum entanglement. As discussed in Sec. III, product and separable numerical ranges coincide for Hermitian operators. Thus, the set of entanglement witnesses consists of Hermitian operators W, such that Tr $W\rho \geq 0$ for all separable ρ and there exists an entangled state σ, such that Tr $W\sigma < 0$. The set of separable quantum states can thus be characterized by a suitably chosen set of entanglement witnesses. Such an approach was advocated in a recent work by Sperling and Vogel,[33] in which various methods for obtaining the minimal product value $\lambda_{\min}^{\otimes}$ of Hermitian matrices were analyzed.

Bound 9 implies that the spectrum of an entanglement witness for any state of a $K \times K$ system has at most $(K-1)^2$ negative eigenvalues, in accordance with recent results of Sarbicki.[34] In the simplest case of $K=2$, one recovers the known statement that any nontrivial entanglement witness in the two-qubit system has exactly one negative eigenvalue.[35]

Our study of product numerical range of a Hermitian operator can thus be directly applied to the positivity problem. For instance, consider the family of one-qubit maps described by the dynamical matrix $D = D(a,b,c)$ defined in (22). It is clear that these matrices are block-positive if and only if $G \leq 1/2$. Therefore, expression (24) for $G = G(a,b,c)$ gives us explicit constraints under which the map corresponding to $D(a,b,c)$ is positive. If this map is not completely positive, the matrix D can be used as a witness of quantum entanglement.

In the above case corresponding to maps acting on two dimensional Hilbert space $\mathcal{H}_2$ any 2-positive map is completely positive. This is a consequence of the theorem of Choi,[31] which implies that if a map acting on K dimensional Hilbert space is K positive, it is also completely positive. Thus, for maps acting on a K-dimensional system, it is interesting to study k-positivity for $k=1$ (equivalent to positivity), $k=2, \ldots K-1$ and $k=K$ (complete positivity). In general, k-block-positive matrices are related to k-positive maps. We are thus in a position to formulate the generalized Jamiołkowski–Choi theorem[29,36] making use of the concept of the restricted numerical range.

Proposition 4: Let Φ be a linear map taking operators on $\mathcal{H}_K$ to operators on $\mathcal{H}_M$. Then

$$\Phi \ \ is \ \ k-positive \Leftrightarrow \Lambda^{(k)}(D_\Phi) \subset [0,\infty). \tag{40}$$

As we explain in Sec. IV B, a special case of Proposition 4 for $k=2$ is of relevance to the distillability problem for quantum states.

B. *n*-copy distillability of a quantum state

It has been known for a long time[37] that bipartite states with distillable entanglement are closely related to 2-positive maps and hence to 2-block positive operators (cf. also Refs. 29 and 38). The precise relation between distillability and 2-block-positivity is the following. Let ρ be an arbitrary state on a bipartite space $\mathcal{H}_N = \mathcal{H}_K \otimes \mathcal{H}_M$. Assume that we allow only local operations and classical communication (LOCC) operations on a single copy of ρ. The state can be distilled into a maximally entangled state only if the partial transpose $(\mathbb{1} \otimes T)\rho$ is *not* a 2-block positive operator, i.e., it is not positive on states with Schmidt rank 2. Otherwise, ρ is one-copy undistillable. Writing this in terms of k-entangled numerical ranges (cf. Sec. III A), we get the following proposition.

Proposition 5: A state with a density matrix ρ on a bipartite space $\mathcal{H}_N = \mathcal{H}_K \otimes \mathcal{H}_M$ is one-copy undistillable if and only if the 2-entangled numerical range of its partial transpose is contained in the nonnegative semiaxis, $\Lambda^{(2)}((\mathbb{1} \otimes T)\rho) \in [0;+\infty)$.

If a state ρ turns out to be one-copy undistillable, it is still possible that a number of copies of ρ can be used for entanglement distillation. Proposition 5 is easily generalized to that situation.

Proposition 6: Let ρ correspond to a state on a bipartite space $\mathcal{H}_N = \mathcal{H}_K \otimes \mathcal{H}_M$. For any integer n, the state is n-copy undistillable if and only if the 2-entangled numerical range of $(\mathbb{1}$

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$\otimes T)\rho^{\otimes n}$ is contained in the nonnegative semiaxis, $\Lambda^{(2)}((\mathbb{1}\otimes T)\rho^{\otimes n})\in[0;+\infty)$.

The symbol $\Lambda^{(2)}$ in Proposition 6 refers to positivity on states of Schmidt rank 2, where the Schmidt rank is calculated with respect to the splitting $\mathcal{H}_N^{\otimes n}=\mathcal{H}_K^{\otimes n}\otimes\mathcal{H}_M^{\otimes n}$ of the multipartite space. This is important to notice because many different splittings of $\mathcal{H}_N^{\otimes n}$ into a tensor product of two factors are possible. Evidently, Proposition 6 is nothing but Proposition 5 applied to $\rho^{\otimes n}$ in place of ρ. This is easy to understand because the tensor product $\rho^{\otimes n}$ represents a number n of identical, independent copies of the state ρ, e.g., coming from a source that produces ρ.

It is natural to mention here a fundamental question concerning distillability of quantum states. Using the language of numerical ranges, we can formulate the problem in the following way.

Given a density matrix ρ on a bipartite Hilbert space $\mathcal{H}_N^{\otimes n}=\mathcal{H}_K^{\otimes n}\otimes\mathcal{H}_M^{\otimes n}$, such that $\Lambda((\mathbb{1}\otimes T)\rho)\notin[0;+\infty)$, can we infer that $\Lambda^{(2)}((\mathbb{1}\otimes T)\rho^{\otimes n})\notin[0;+\infty)$ for some positive integer n?

In other words, is a bipartite state ρ with a negative partial transpose always distillable, possibly using a huge number n of copies of ρ? This question has not yet been answered, despite a considerable effort and some partial results (cf., e.g., Ref. 39).

C. Minimum output entropy and product numerical range

Consider a completely positive map Φ acting on the set Ω_N of normalized quantum states of dimension N. Minimum output entropy (see, e.g., Ref. 40, Chap. 7) is defined as

$$S_{\min}(\Phi)=\min_{\rho}\{S(\Phi(\rho))\}, \tag{41}$$

with $\rho\in\Omega_N$. Since the von Neumann entropy is concave, the minimum is attained on the boundary and thus

$$S_{\min}(\Phi)=\min_{|\psi\rangle\langle\psi|}\{S(\Phi(|\psi\rangle\langle\psi|))\}, \tag{42}$$

where $|\psi\rangle\langle\psi|\in\Omega_N$ are pure states. Therefore, the minimum output entropy can be interpreted as a certain measure of decoherence introduced by the channel.

In Refs. 41 and 42, it was proven that minimum output entropy (and thus Holevo capacity) is additive for unital channels. It is now known, however, that minimum output entropy is not additive in the general case.[43] Here we provide a characterization of the minimum output entropy for one-qubit channels using product numerical range of the dynamical matrix.

Proposition 7: Let Φ be a completely positive, trace preserving (CP-TP) map acting on Ω_2. Then

$$S_{\min}(\Phi)=\lambda\log(\lambda)+(1-\lambda)\log(1-\lambda), \tag{43}$$

where λ is a minimal value of product numerical range for the dynamical matrix D_Φ,

$$\lambda=\lambda_{\min}^{\otimes}(D_\Phi). \tag{44}$$

Proof: Let us define $f(x)=-x\log_2(x)-(1-x)\log_2(1-x)$, which is increasing for $x\in[0,\frac{1}{2}]$. Directly from the definition of minimum output entropy we can write

$$S_{\min}(\Phi)=\min_{|i\rangle}S(\Phi(|i\rangle\langle i|))=\min_{|i\rangle}f(\lambda_{\min}(\Phi(|i\rangle\langle i|)))=f(\min_{|i\rangle}\lambda_{\min}(\Phi(|i\rangle\langle i|))). \tag{45}$$

Now since $\langle k|\Phi(|i\rangle\langle j|)|l\rangle=\langle k\otimes i|D_\Phi|l\otimes j\rangle$ [see Ref. 13, Eq. 11.25], we can rewrite the above expression as

$$S_{\min}(\Phi)=f(\min_{|i\rangle,|j\rangle}\langle j|\Phi(|i\rangle\langle i|)|j\rangle)=f(\min_{|i\rangle,|j\rangle}\langle j\otimes i|D_\Phi|j\otimes i\rangle)=f(\lambda_{\min}^{\otimes}(D_\Phi)). \tag{46}$$

■

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Using the above proposition, we can easily calculate minimal output entropy for channels listed below.

First we consider the amplitude damping, phase damping, phase flip, bit-flip, and bit-phase flip channels. In all those cases we can see from the Kraus form that the spectrum of the dynamical matrix has two zero eigenvalues. Then the plane spanned by the two eigenvectors corresponding to the zero eigenvalue contains at least one product state.[25] Thus, $\lambda_{\min}^{\otimes}(D_\Phi)=0$ and the minimum output entropy for this channels is equal to zero. Using Proposition 7 one can also easily calculate minimum output entropy for some other one qubit channels. Consider the Werner–Holevo channel, described by the following dynamical matrix:

$$D_{\Phi_{\mathrm{HW}}}=\begin{pmatrix} \frac{p+1}{2} & 0 & 0 & 0 \\ 0 & \frac{1-p}{2} & p & 0 \\ 0 & p & \frac{1-p}{2} & 0 \\ 0 & 0 & 0 & \frac{p+1}{2} \end{pmatrix} \tag{47}$$

for $p\in[-1,1/3]$. In this case $\lambda_{\min}^{\mathrm{loc}}(D_{\Phi_{\mathrm{HW}}})=\frac{1}{2}(1-|p|)$ and thus

$$S_{\min}(\Phi_{\mathrm{HW}})=-\frac{1}{2}(1-|p|)\log_2\frac{1}{2}(1-|p|)-\frac{1}{2}(1+|p|)\log_2\frac{1}{2}(1+|p|) \tag{48}$$

$$=-\frac{\log\left(\frac{1}{4}-\frac{p^2}{4}\right)+2p\tanh^{-1}(p)}{\log(4)}. \tag{49}$$

In the case of higher dimensional quantum channels, we can use properties of product numerical range to check, whether for a given channel its minimal output entropy is equal zero.

Proposition 8: For any CP-TP map we have

$$S_{\min}(\Phi)=0 \quad iff \quad 1\in\Lambda^{\otimes}(D_\Phi). \tag{50}$$

Proof: Since $1\in\Lambda^{\otimes}(D_\Phi)$, there exists $|i\rangle$, $|j\rangle$, such that

$$1=\langle i\otimes j|D_\Phi|i\otimes j\rangle=\langle i|\Phi(|j\rangle\langle j|)|i\rangle. \tag{51}$$

Because Φ is CP-TP channel, we have tr $\Phi(|j\rangle\langle j|)=1$ and thus

$$\Phi(|j\rangle\langle j|)=|i\rangle\langle i|. \tag{52}$$

The proposition follows. ■

D. Local discrimination of unitary operators

The problem of local distinguishability of multipartite quantum states was analyzed by Walgate *et al.*[44] Following their work, Duan *et al.*[16] have shown that two unitary operations U_1 and U_2 are locally distinguishable if and only if $0\in\Lambda^{\otimes}(V)$, where $V=U_1^{\dagger}U_2$. If this is the case, then there exists a product state $|\psi\rangle=|\phi_A,\phi_B\rangle$, such that the states $U_1|\psi\rangle$ and $U_2|\psi\rangle$ are orthogonal and thus distinguishable.

Our results on product numerical range allow us to solve the problem of local distinguishability for a wide class of unitary operators. If the operator $V=U_1^{\dagger}U_2$ has the tensor product

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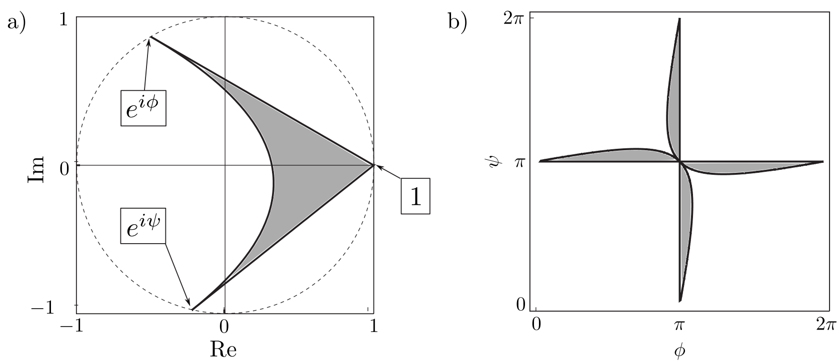

FIG. 7. (a) Product numerical range for matrix (53) with $\phi=2\pi/3$ and $\psi=10\pi/7$. (b) The region in the space of parameters (ϕ,ψ) corresponding to locally distinguishable pairs (U_1,U_2) when $U_1^\dagger U_2=V(\phi,\psi)$.

structure, $V=V_1\otimes V_2$, the two unitaries U_1, U_2 are distinguishable if and only if the numerical range of any of the factors V_1, V_2 contains zero. This is the case when 0 belongs to the convex hull of the spectrum of the factor V_1 or of the factor V_2.

Let us now deal with a more general case of V without the tensor product structure. Consider, for instance, a family of unitary matrices of order of 4,

$$V(\phi,\psi)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & e^{i\phi} & 0 & 0\\ 0 & 0 & e^{i\psi} & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \tag{53}$$

for $\phi,\psi\in[0,2\pi]$.

It is easy to show that the product numerical range of V is a bounded region of $\mathbb{C}$ whose border consists of the segments $[e^{i\phi},1]$, $[1,e^{i\psi}]$ and the line

$$\gamma:[0,1]\ni t\mapsto t^2e^{i\phi}+(1-t)^2e^{i\psi}+2t(1-t)\in\mathbb{C}. \tag{54}$$

For example, Fig. 7(a) shows the shape of the product numerical range of $V\left(\frac{2\pi}{3},\frac{10\pi}{7}\right)$.

Using Eq. (54), it is not difficult to check for which values of the phases ϕ and ψ the product numerical range of $V(\phi,\psi)$ contains 0, so any U_1 and U_2, such that $U_1^\dagger U_2=V(\phi,\psi)$ are locally distinguishable. Figure 7(b)) shows, in gray, the set of parameters (ϕ,ψ) corresponding to such distinguishable pairs (U_1,U_2). Explicitly, we have

$$\begin{aligned}0\in\Lambda^{\otimes}(V(\phi,\psi))\Leftrightarrow\{&|\sin\psi|\cos\phi+|\sin\phi|\cos\psi+2\sqrt{|\sin\phi\sin\psi|}\leq 0\wedge\sin\phi\sin\psi\\ &\leq 0\wedge(\phi,\psi)\notin\{(0,0),(2\pi,2\pi)\}\}.\end{aligned} \tag{55}$$

For any two unitary matrices U_1 and U_2, such that $V=U_1^\dagger U_2$ satisfies the above constraints, it is possible to find a product state $|\chi,\xi\rangle$ with the property $\langle\chi,\xi|V|\chi,\xi\rangle=0$. A detailed construction of this state, presented in Appendix A, allows one to design the scheme of local discrimination between the unitary gates U_1 and U_2.

E. Local fidelity and entanglement measures

Several tasks of quantum information processing relay on the ability to approximate a given quantum state ϱ_1 by some other state ϱ_2. Alternatively, one attempts to distinguish ϱ_1 from ϱ_2. To characterize both problems quantitatively, one may use *fidelity*, which can be interpreted as a "transition probability" in the space of quantum states,[45]

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$$F(\varrho_1,\varrho_2) = [\mathrm{Tr}|\sqrt{\varrho_1}\sqrt{\varrho_2}|]^2. \tag{56}$$

We are going to follow here the original definition by Jozsa,[46] but one has to be warned that some later articles use the name "fidelity" for $\sqrt{F}$. If one of the states is pure, $\varrho_1=|\psi_1\rangle\langle\psi_1|$, formula (56) simplifies and $F=\langle\psi_1|\varrho_2|\psi_1\rangle$. Thus, in this case fidelity has a simple interpretation of *probability* that the state ϱ_2 is projected onto a pure state $|\psi_1\rangle$.

Consider two arbitrary mixed states ϱ_1 and ϱ_2 acting on a Hilbert space $\mathcal{H}_N$. Although fidelity between these states is fixed and given by (56), one may pose a question to what extent fidelity can grow if local unitary operations are allowed. In other words, one asks about the fidelity between ϱ_1 and $U\varrho_2U^\dagger$ maximized over all unitaries $U\in U(N)$. This problem was studied in Ref. 47, where the following bounds were established:

$$F(p^\uparrow,q^\downarrow) \le F(\varrho_1,U\varrho_2U^\dagger) \le F(p^\uparrow,q^\uparrow) = F(p^\downarrow,q^\downarrow). \tag{57}$$

The vectors p and q represent the spectra of ϱ_1 and ϱ_2, while the up/down arrows indicate that the eigenvalues are put in the nondecreasing (nonincreasing) order. Arguments of the fidelity in the above equation denote thus diagonal matrices which represent classical states.

In this section we analyze an analogous problem for multipartite systems: What maximum fidelity between two given states of such a system can be achieved, if arbitrary local unitary operations are allowed? We provide a solution of this problem in the special case when both quantum states are pure and derive bounds for the local fidelity in the case where ρ is a diagonal mixed state.

Let $|\phi\rangle$ be a vector and ϱ an arbitrary mixed state, both on $\mathcal{H}_{AB}=\mathcal{H}_A\otimes\mathcal{H}_B$. For simplicity we will restrict our attention to the symmetric case and assume that $\dim(\mathcal{H}_A)=\dim(\mathcal{H}_B)=N$.

The fidelity of a mixed state with respect to a pure state is given by an expectation value, $F=\langle\phi|\varrho|\phi\rangle$. We are going to study the question to what extend this quantity can be increased by applying arbitrary local unitary operations $U_A\otimes U_B$. In other words, we look for the *local fidelity* defined as the maximum,

$$F^{\max}(\varrho,\phi) = \max_{U_A\otimes U_B}\langle\phi|(U_A\otimes U_B)^\dagger\varrho(U_A\otimes U_B)|\phi\rangle. \tag{58}$$

It is instructive to relate this quantity to a generalized numerical radius of an operator X, defined as the largest modulus of an element of its numerical range. Similarly for an operator X acting on a composed Hilbert space, one defines product numerical radius as the largest modulus of an element of $\Lambda^\otimes(X)$. This notion can be further generalized, and for any operator X and an auxiliary operator C acting on the Hilbert space $\mathcal{H}_N=\mathcal{H}_K\otimes\mathcal{H}_M$, one defines the C-product numerical radius,[17]

$$r_C^\otimes(X) = \max\{|z|:z=\mathrm{tr}(U_1\otimes U_2)X(U_1\otimes U_2)^\dagger C,\ \ U_1\in U(K),\ \ U_2\in U(M)\}, \tag{59}$$

and other notions listed in Table II. The problem of finding the local fidelity is then equivalent to determining the C-product numerical radius of the operator $X=|\phi\rangle\langle\phi|$ for $C=\varrho$.

Let us first solve the problem in the special case where the analyzed state is pure, $\varrho=|\psi\rangle\langle\psi|$. It is then useful to represent both pure states using their Schmidt decompositions (37),

$$|\phi\rangle = \sum_{i=1}^{N}\sqrt{\lambda_i}|i\rangle\otimes|i\rangle, \quad |\psi\rangle = \sum_{j=1}^{N}\sqrt{\mu_j}|j\rangle\otimes|j\rangle. \tag{60}$$

The vector λ of Schmidt coefficients set in a decreasing (increasing) order will be denoted by $\lambda^\downarrow$ and $\lambda^\uparrow$, respectively. This notation allows one to formulate the following lemma.

Lemma 1: For arbitrary local unitary operation $U_A\otimes U_B$ and pure states $|\phi\rangle$, $|\psi\rangle$, one has

$$0 \le |\langle\psi|U_A\otimes U_B|\phi\rangle|^2 \le F(\mu^\downarrow,\lambda^\downarrow) = \left(\sum_{j=1}^{N}\sqrt{\lambda_j^\downarrow\mu_j^\downarrow}\right)^2. \tag{61}$$

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The lower bound is a trivial consequence of the definition of fidelity. The upper bound follows from the theorem of Uhlmann which states that fidelity is given by the maximal overlap between purifications of both states, and the bound in Ref. 47, Eq. (4.19). This result follows also from the recent work of Schulte-Herbrüggen *et al.* (Ref. 18, Proposition IV.1).

If one of the states is separable, $|\phi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$, its Schmidt vector has only a single nonvanishing component, $\lambda^{\downarrow} = (1,0,\ldots,0)$, so overlap (61) is bounded by the largest Schmidt coefficient $\mu_{\max}$ of the state $|\psi\rangle$. This is a special case of the *geometric measure of entanglement* of a multipartite state $|\psi\rangle$, defined as the logarithm of the maximum projection on any product state,[48]

$$E_g(|\psi\rangle) = -\log(\max_{U_{\mathrm{loc}}} |\langle\psi| U_1 \otimes U_2 \otimes \cdots \otimes U_m |0,\ldots 0\rangle|^2). \tag{62}$$

Here $|0,\ldots 0\rangle$ represents an arbitrary product state, so transforming it by a local unitary matrix, one explores the entire set of separable pure states of the m-partite system. Observe that the argument of the logarithm in the above expression is just equal to the product numerical radius of the projector onto the analyzed state, $X = |\psi\rangle\langle\psi|$.

In recent papers,[49,50] it was shown that the above maximization procedure becomes simpler if the multipartite state $|\psi\rangle$ is symmetric with respect to permutations of the subsystems, and all its coefficients in the product basis are non-negative. Then the maximum in (62) is achieved for the tensor product of a single unitary matrix, $U_{\mathrm{loc}} = U^{\otimes m}$, so the search for $E_g(\psi)$ can be reduced to the space of a smaller dimension. It is then natural to ask whether this observation can be generalized for the problem of determining the product numerical radius of any multipartite Hermitian operator X, provided that X is symmetric with respect to permutations and it satisfies suitable positivity conditions. This problem was considered in a very recent paper by Hübner *et al.*[51]

Thus, the product numerical radius is useful in characterizing quantum entanglement of a pure state of a multipartite system. Interestingly, the product C-numerical radius of a Hermitian biconcurrence matrix introduced by Badziag *et al.*[52] can be applied to describe the degree of quantum entanglement for any mixed state of a bipartite system.

Let us then return to the bipartite problem and discuss the case when one of the two states in expression (58) for local fidelity is pure while the other is mixed. Assume that the pure state $|\phi\rangle$ is given by its Schmidt decomposition (60), while ϱ is a diagonal mixed state, $\varrho = \sum_{ij=1}^{N} p_{ij} |i\rangle\langle i| \otimes |j\rangle\langle j|$. The maximal local fidelity between these states can be bounded by the following lemma, proven in Appendix B.

Lemma 2: The maximal fidelity between a pure state ψ and diagonal state ρ is bounded from above,

$$\max_{U,V \in U(N)} F((U\otimes V)|\psi\rangle\langle\psi|(U\otimes V)^{\dagger}, \varrho) \le \max \sum_{ij=1}^{N} p_{ij} B_{ij}, \tag{63}$$

where the maximum on the right-hand side is taken over all collections of non-negative real numbers B_{ij} that satisfy the constraints, for any $\{i_1, i_2, \ldots, i_r\} \subset \{1,2,\ldots,N\}$,

$$\sum_{j=1}^{N} [B_{i_1 j} + B_{i_2 j} + \ldots + B_{i_r j}] \in \left[\sum_{k=1}^{r} \lambda_{(k)}, \sum_{k=N-r+1}^{N} \lambda_{(k)} \right], \tag{64}$$

and for any $\{j_1, j_2, \ldots, j_s\} \subset \{1,2,\ldots,N\}$,

$$\sum_{i=1}^{N} [B_{i j_1} + B_{i j_2} + \ldots + B_{i j_s}] \in \left[\sum_{k=1}^{s} \lambda_{(k)}, \sum_{k=N-s+1}^{N} \lambda_{(k)} \right], \tag{65}$$

where $\lambda_{(1)}, \lambda_{(2)}, \ldots, \lambda_{(N)}$ are Schmidt coefficients of ψ in ascending order.

The maximum on the right-hand side in Lemma 2 is attained at the edges of the polygon defined by constraints (64) and (65). The bounds obtained in this way can be easily computed numerically using the simplex algorithm.

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F. Local dark spaces and error correction codes

Consider a quantum operation Φ acting in the space of mixed quantum states of size N, which can be represented in the Kraus form,

$$\rho' = \Phi(\rho) = \sum_{i=1}^{M} Y_i \rho Y_i^\dagger. \tag{66}$$

To assure that the trace is preserved by the operation, the set of M Kraus operators has to satisfy an identity resolution, $\Sigma_{i=1}^{M} Y_i^\dagger Y_i = \mathbb{1}$.

Consider a l-dimensional subspace $P_l = \Sigma_{i=1}^{l} |i\rangle\langle i|$ embedded in $\mathcal{H}_N$. If it satisfies the set of M conditions

$$P_l X_m P_l = \lambda_m P_l \quad \text{for } m = 1, \dots, M, \tag{67}$$

where $X_m = Y_m^\dagger Y_m$ and $\lambda_m \in \mathbb{C}$ no information goes outside of this subspace,[53] so P_l is called a *dark subspace*.[54]

If a subspace P_k fulfils even stronger conditions of type (67),

$$P_l Y_i^\dagger Y_j P_l = \lambda_{ij} P_l \quad \text{for } i,j = 1, \dots, M, \tag{68}$$

then quantum information stored in the system can be recovered, so the subspace P_l provides an error correction code.[55,56] Note that P_l has to simultaneously satisfy all the M^2, Eqs. (68). The complex numbers λ_{ij} corresponding to different X_{ij}'s may be different.

From algebraic perspective condition (67) implies that λ_m belongs to the numerical range of order l of the operator X_m.[57] In full analogy to the product numerical range, one may introduce the concept of *product numerical range of higher rank* as defined in Table II. This notion can be used to identify dark spaces or error correction codes with a local structure.[58] The distinguished subspace $P_l^{\otimes}$, which solves the set of Eqs. (66), can be chosen to be in the product form, $P_l^{\otimes} = \Sigma_{i=1}^{l} |i \otimes i\rangle\langle i \otimes i|$.

V. CONCLUDING REMARKS

In this work we investigated basic properties of numerical range of an operator restricted to some class of quantum states. In particular, we analyzed the case of operators acting on a Hilbert space with a tensor product structure, often used to describe composed quantum systems. In this case one defines the product numerical range of an operator. We reviewed basic properties of this notion and presented some examples of operators for which product numerical range can be found analytically.

To tackle the problem in a general case, however, we had to rely on numerical computations. In particular, we investigated an ensemble of $N=4$ random density matrices distributed according to the Hilbert–Schmidt measure and compared the probability distributions of both edges of the product range with probability distributions for individual eigenvalues.

In the case of a non-Hermitian operator, its product numerical range forms a connected set in the complex plane. In general, this set is not convex. The product numerical range of an operator acting on a twofold tensor product is simply connected. However, this property does not hold for operators acting on a space with a larger number of subsystems. For any operator with a tensor product structure, its product range is equal to the Minkowski product of numerical ranges of all factors. The theory of the Minkowski product of various sets in the complex plane, recently developed by Farouki *et al.*,[23] can thus be directly applied to characterize the product numerical range of operators of the tensor product form. In this way we managed to establish product numerical range of a unitary product matrix $U^{\otimes n}$.

Numerical range can also be generalized by taking other restrictions on the set of quantum states. Although we studied here the case of numerical range restricted to separable and k-entangled states, one may also use other restricted sets of quantum states or combine these conditions, analyzing, for instance, the set of real product states. As the product states of the K

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$\times M$ system can also be considered as coherent states with respect to the composite group $SU(K)\otimes SU(M)$,[59] an analogous relation holds for the corresponding numerical ranges.

Numerical range can also be generalized in other direction: for each case of a restricted numerical range, one can introduce concepts and generalizations known for the standard numerical range. In Table II we have collected standard definitions of numerical range, numerical radius, C-numerical range, and higher rank numerical range,[57] along with their counterparts defined for Hilbert space of the form of an m-fold tensor product, $\mathcal{H}_N=\mathcal{H}_{n_1}\otimes\cdots\otimes\mathcal{H}_{n_m}$, with $N=n_1\ldots n_m$. Note that C-numerical range as well as product C-numerical range reduce to the numerical range (product numerical range) for $C=\mathrm{diag}(\{1,0,\ldots,0\})$ and this case was already analyzed in Ref. 15.

Observe that the above concepts arise naturally in a variety of problems in quantum information theory. For instance, being in a position to find the product numerical range of an arbitrary operator, one could advance fundamental problems concerning the characterization of the set of positive maps or description of the set of entangled states and finding the minimum output entropy of a one-qubit quantum channel. Therefore, improving techniques of finding restricted numerical ranges would have direct implications for the theory of quantum information and quantum control. For example, in this work we have established the positivity of a certain family of one-qubit maps, we solved the problem of local distinguishability between a class of two-qubit unitary gates and analyzed the properties of local fidelity between quantum states.

In conclusion, we advocate further studies on restricted numerical range and cognate concepts. On one hand, the restricted numerical range is an interesting subject for mathematical investigations. On the other hand, it proves to be a versatile algebraic tool, useful in tackling various problems of quantum theory.

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APPENDIX A: PRODUCT VECTORS FOR LOCAL DISCRIMINATION BETWEEN U_1 AND U_2

The discussion in Sec. IV D left aside the question of precisely how the unitaries U_1, U_2 fulfilling $\Lambda^{\otimes}(U_1^{\dagger}U_2)=0$ can be distinguished. To accomplish this task in practice, one needs to find a product vector $|\chi,\xi\rangle$, such that $\langle\chi,\xi|U_1^{\dagger}U_2|\chi,\xi\rangle=0$. There exists, in general, an infinite number of such vectors. In the case $U_1^{\dagger}U_2=V(\phi,\psi)$ analyzed in Sec. IV D, it is not difficult to find all of them. Recall that $V(\phi,\psi)$ is of the diagonal form $\mathrm{diag}(1,e^{i\phi},e^{i\psi},1)$ with respect to the tensor product basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ of $\mathcal{H}_2\otimes\mathcal{H}_2$. Let us write $|\xi\rangle=\sqrt{t}e^{i\kappa_0}|0\rangle+\sqrt{1-t}e^{i\kappa_1}|1\rangle$ and $|\chi\rangle=\sqrt{s}e^{i\delta_0}|0\rangle+\sqrt{1-s}e^{i\delta_1}|1\rangle$ for $s,t\in[0,1]$ and κ_0, κ_1, δ_0, δ_1 arbitrary real numbers. Thus, we assume that $|\chi\rangle$ and $|\xi\rangle$ are of unit norm, which is permissible. It is now easy to see that

$$\langle\chi,\xi|V(\phi,\psi)|\chi,\xi\rangle = ts + (1-t)(1-s) + e^{i\phi}t(1-s) + e^{i\psi}(1-t)s, \tag{A1}$$

where $s,t\in[0,1]$.

Formula (A1) gives us some idea of how the results presented in Sec. IV D were obtained. Note that the phases κ_0, κ_1, δ_0 and δ_1 are irrelevant to the value of $\langle\chi,\xi|V(\phi,\psi)|\chi,\xi\rangle$. Thus, any product vector that fulfils certain relations between the amplitudes t and s can be used for perfect discrimination between the two unitaries. Note that this is a general property whenever $U_1^{\dagger}U_2$ is diagonal with respect to some tensor product basis of $\mathcal{H}_K\otimes\mathcal{H}_M$ and $0\in\Lambda^{\otimes}(U_1^{\dagger}U_2)$.

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In order to solve Eq. (A1) for s and t, we first observe that

$$\mathrm{Im}(\langle\chi,\xi|V(\phi,\psi)|\chi,\xi\rangle)=0 \tag{A2}$$

reduces to $\sin\phi t(1-s)+\sin\psi s(1-t)=0$ or

$$s=\frac{t\sin\phi}{t(\sin\phi+\sin\psi)-\sin\psi}. \tag{A3}$$

If we substitute this in Eq. (A1), we get the condition $\langle\chi,\xi|V(\phi,\psi)|\chi,\xi\rangle=0$ in the following form:

$$t^2\sin\phi+(1-t)(t\sin(\phi-\psi)-(1-t)\sin\psi)=0. \tag{A4}$$

We can solve (A4) for $t\in[0,1]$ under the assumption that $0\in\Lambda^{\otimes}(V(\phi,\psi))$ [cf. the conditions on the right-hand side of (55)]. The result is

$$t=\frac{\sqrt{\sin(\phi-\psi)^2+4\sin\phi\sin\psi}+|\sin(\phi-\psi)|+2|\sin\psi|}{2(|\sin\phi|+|\sin(\phi-\psi)|+|\sin\psi|)}. \tag{A5}$$

By symmetry we obtain an expression for s,

$$s=\frac{\sqrt{\sin(\psi-\phi)^2+4\sin\psi\sin\phi}+|\sin(\psi-\phi)|+2|\sin\phi|}{2(|\sin\psi|+|\sin(\psi-\phi)|+|\sin\phi|)}. \tag{A6}$$

Hence the product vector useful for perfect local discrimination between U_1 and U_2 can be any of the family

$$(\sqrt{t}e^{i\kappa_0}|0\rangle+\sqrt{1-t}e^{i\kappa_1}|1\rangle)\otimes(\sqrt{s}e^{i\delta_0}|0\rangle+\sqrt{1-s}e^{i\delta_1}|1\rangle), \tag{A7}$$

with $\kappa_0,\ \kappa_1,\ \delta_0,\ \delta_1\in\mathbb{R}$ and s,t given by formulas (A6) and (A5), respectively. This only works when $U_1^{\dagger}U_2=V(\phi,\psi)$ and $0\in\Lambda^{\otimes}(U_1^{\dagger}U_2)$.

APPENDIX B: PROOF OF LEMMA 2

Let us introduce matrix A which depends on the vector λ and a local unitary matrix $U\otimes V$, with entries

$$A_{ij}=\left|\sum_{k=1}^{N}\sqrt{\lambda_k}\langle U(k)|i\rangle\langle V(k)|j\rangle\right|^2, \tag{B1}$$

where by $U(k)$ we mean $U(|k\rangle)$, so that $\langle U(k)|i\rangle$ corresponds to $\langle k|U^{\dagger}|i\rangle$ in the usual physicists' notation. Similarly, $\langle k|U(i)\rangle=\langle k|U|i\rangle$.

Using the notation of Eq. (B1), we arrive at a handy expression for the expectation value,

$$\langle\psi|(U\otimes V)^{\dagger}\rho(U\otimes V)|\psi\rangle=\sum_{ij=1}^{N}p_{ij}A_{ij}, \tag{B2}$$

which we wish to maximize over the set of local unitaries. The first thing to notice is that A_{ij} are non-negative real numbers and

$$\sum_{ij=1}^{N}A_{ij}=1, \tag{B3}$$

thus the matrix A treated as vector is an element of standard (N^2-1)-simplex.

Matrix A defined above satisfies the following lemma.

Lemma 3: For any $\{i_1,i_2,\ldots,i_r\}\subset\{1,2,\ldots,N\}$,

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$$\sum_{j=1}^{N}[A_{i_1 j}+A_{i_2 j}+\cdots+A_{i_r j}]\in[\lambda_{(1)}+\lambda_{(2)}+\cdots+\lambda_{(r)},\lambda_{(N)}+\lambda_{(N-1)}+\cdots+\lambda_{(N-r+1)}], \tag{B4}$$

and for any $\{j_1,j_2,\ldots,j_s\}\subset\{1,2,\ldots,N\}$,

$$\sum_{i=1}^{N}[A_{ij_1}+A_{ij_2}+\cdots+A_{ij_s}]\in[\lambda_{(1)}+\lambda_{(2)}+\cdots+\lambda_{(s)},\lambda_{(N)}+\lambda_{(N-1)}+\cdots+\lambda_{(N-s+1)}], \tag{B5}$$

where $\lambda_{(1)},\lambda_{(2)},\ldots,\lambda_{(N)}$ *are the Schmidt coefficients of* $|\psi\rangle$ *sorted ascendingly.*

Proof: First we write

$$\begin{aligned}\sum_{j=1}^{N}[A_{i_1 j}+A_{i_2 j}+\cdots+A_{i_r j}]&=\sum_{j=1}^{N}\left[\left|\sum_{k=1}^{N}\sqrt{\lambda_k}\langle Uk|i_1\rangle\langle V(k)|j\rangle\right|^2+\cdots+\left|\sum_{k=1}^{N}\sqrt{\lambda_k}\langle U(k)|i_r\rangle\langle V(k)|j\rangle\right|^2\right]\\&=\sum_{j=1}^{N}\left[\sum_{k_1 l_1=1}^{N}\sqrt{\lambda_{k_1}\lambda_{l_1}}\langle U(k_1)|i_1\rangle\langle V(k_1)|j\rangle\langle i_1|U(l_1)\rangle\langle j|V(l_1)\rangle+\cdots\right.\\&\quad\left.+\sum_{k_r l_r=1}^{N}\sqrt{\lambda_{k_r}\lambda_{l_r}}\langle U(k_r)|i_r\rangle\langle V(k_r)|j\rangle\langle i_r|U(l_r)\rangle\langle j|V(l_r)\rangle\right]\\&=\sum_{k_1 l_1=1}^{N}\sqrt{\lambda_{k_1}\lambda_{l_1}}\langle U(k_1)|i_1\rangle\langle i_1|U(l_1)\rangle\sum_{j=1}^{N}\langle V(k_1)|j\rangle\langle j|V(l_1)\rangle+\cdots\\&\quad+\sum_{k_r l_r=1}^{N}\sqrt{\lambda_{k_r}\lambda_{l_r}}\langle U(k_r)|i_r\rangle\langle i_r|U(l_r)\rangle\sum_{j=1}^{N}\langle V(k_r)|j\rangle\langle j|V(l_r)\rangle.\end{aligned} \tag{B6}$$

Since $|j\rangle$ form a basis, we have the identity $\Sigma|j\rangle\langle j|=\mathbb{1}$. Thus,

$$\begin{aligned}\sum_{j=1}^{N}[A_{i_1 j}+A_{i_2 j}+\cdots+A_{i_r j}]&=\sum_{k_1 l_1=1}^{N}\sqrt{\lambda_{k_1}\lambda_{l_1}}\langle U(k_1)|i_1\rangle\langle i_1|U(l_1)\rangle\langle V(k_1)|V(l_1)\rangle+\cdots\\&\quad+\sum_{k_r l_r=1}^{N}\sqrt{\lambda_{k_r}\lambda_{l_r}}\langle U(k_r)|i_r\rangle\langle i_r|U(l_r)\rangle\langle V(k_r)|V(l_r)\rangle=\sum_{k_1=1}^{N}\lambda_{k_1}|\langle U(k_1)|i_1\rangle|^2\\&\quad+\cdots+\sum_{k_r=1}^{N}\lambda_{k_r}|\langle U(k_r)|i_r\rangle|^2.\end{aligned} \tag{B7}$$

This fact, combined with Corollary 4.3.18 of Horn and Johnson,[60] proves the lemma. ■

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