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Sterowalność układów z opóźnieniami

Streszczenie

W referacie rozpatrywane są zagadnienia sterowalności liniowych, ciągłych, skończenie wymiarowych układów dynamicznych ułamkowego drugiego rzędu z pojedynczym skupionym opóźnieniem we współrzędnych stanu oraz rozłożonym opóźnieniem w sterowaniach dopuszczalnych. Przedstawiono model matematyczny układu dynamicznego w postaci ułamkowego różniczkowego równania stanu. Podano definicję pochodnej ułamkowego rzędu, stanu chwilowego oraz stanu zupełnego układu dynamicznego z opóźnieniami oraz definicje zbiorów osiągalnych.

W przypadku układów ułamkowych drugiego rzędu rozwiązanie różniczkowego równania stanu jest innej postaci niż w przypadku układów pierwszego rzędu. Ma to istotny wpływ na postać warunków sterowalności.

Następnie przypomniano definicję globalnej względnej sterowalności liniowego układu dynamicznego w zadanym przedziale czasowym dla układów dynamicznych z opóźnieniami. Wykorzystując liniowość oraz stacjonarność rozpatrywanego układu dynamicznego zastosowano do wyznaczenia postaci rozwiązania odwrotne przekształcenie Laplace'a uzyskując w ten sposób analityczne rozwiązanie liniowego, ułamkowego, różniczkowego równania stanu układu dynamicznego.

Wykorzystując metody analizy funkcjonalnej, a w szczególności twierdzenia z zakresu liniowych operatorów w przestrzeniach Hilberta zaproponowano postać macierzy sterowalności, będącej uogólnieniem na przypadek układów dynamicznych z opóźnieniami znanej macierzy sterowalności układów dynamicznych bez opóźnień.

W dalszej części referatu na podstawie zaproponowanej macierzy sterowalności sformułowano oraz udowodniono algebraiczne warunki konieczne i wystarczające globalnej względnej sterowalności w zadanym przedziale czasowym dla rozpatrywanego układu dynamicznego z opóźnieniami. Ponadto przedyskutowano wzajemne relacje zachodzące pomiędzy poszczególnymi rodzajami sterowalności.

W końcowej części referatu przedstawiono możliwe uogólnienia kryteriów sterowalności względnej na przypadek układów dynamicznych z rozłożonymi opóźnieniami zarówno we współrzędnych stanu, jak i w sterowaniach dopuszczalnych.

Controllability of Systems with Delays

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Abstract—**The main purpose of this presentation is to study controllability of linear continuous-time fractional dynamical systems containing both lumped constant delay in state variables and distributed delays in admissible controls. Necessary and sufficient conditions for relative controllability in finite time interval are formulated and proved using theory of linear bounded operators, solution properties of fractional differential equations and results taken directly from linear matrix algebra. The main result of the paper is to show, that global relative controllability of fractional linear systems with different types of delays is equivalent to non-singularity of suitably defined relative controllability matrix.**

Keywords—controllability; linear systems; fractional systems, delayed systems; distributed delays

I. INTRODUCTION

In the literature there are many different definitions of controllability, both for linear and nonlinear or semilinear dynamical systems [4], [5], [15] [18], [22], [25], [29]. Controllability concept strongly depends on class of dynamical control systems and on the set of admissible controls, [10], [12], [17], [32], [33]. Therefore, nonlinear or semilinear fractional systems there exist many different necessary and sufficient conditions for global and local controllability [4], [5], [24].

Controllability of linear systems with different types of delays was considered in many monographs [11], [13], [18], [21], survey papers [19] and [20] and in regular papers [9], [10], [16], [17].

The various types of fractional differential equations have many applications in different fields of technique including for example signal processing, theory of visco-elastic materials [1], [30], supercapacitors [23] filter description and design, circuit theory [13], computer networks, and bioengineering [11].

The main purpose of this paper is to study the relative global controllability of linear fractional second order delay dynamical systems containing both single lumped constant time delay in the state variables and distributed delay in the admissible controls.

This is natural generalizations of controllability concepts, which is rather well known in the theory of finite dimensional linear control systems, without delays in state variables or in admissible controls. Using techniques similar to those presented in monographs [18], and [21] and in the series of papers [12], [16] and [17] we shall formulate and prove necessary and sufficient conditions for global relative controllability of fractional control systems in a prescribed time interval.

This paper is organized as follows: section 2 contains mathematical model of linear, stationary fractional second order stationary dynamical system with multiple time variable point delays in admissible controls. Moreover, in this section basic solution of fractional second order linear finite dimensional differential equation is presented in compact integral form and its properties are also discussed. In section 3 definition of global relative controllability in a given time interval is recalled. Next, using results and methods taken directly from linear functional analysis [31], global relative controllability problem is mathematically stated and considered. Moreover, using suitably defined relative controllability matrix necessary and sufficient conditions for global relative controllability in a finite time interval are formulated and proved. admissible control. Finally, section 4 contains concluding remarks and proposes some open controllability problems for more general fractional systems.

II. SYSTEM DESCRIPTION

Let us consider linear, second order fractional, delay dynamical systems containing single lumped constant delay in the state variables and distributed delays in admissible controls, described by the following fractional differential state equation [2], [3], [11], [24], [25].

$$D^{\alpha}x(t) = Ax(t) + Cx(t-h) + \int_{-h}^{0} d_{\tau}B(t,\tau)u(t+\tau) \tag{1}$$

$$t \in [t_0 - h, t_1]$$

with initial complete state

$$x(t) = x_{t_0}(t) = \varphi(t), \quad u(t) = u_{t_0}(t) \quad for \quad t \in [t_0 - h, t_0] \tag{2}$$

where

$1 < \alpha \le 2$, $D^{\alpha}(t)$ **denotes second order fractional Caputo derivative, defined as follows**

$$\boldsymbol{D^{\alpha}f(t) = \frac{1}{\Gamma(2-\alpha)}\int_{0}^{t}(t-s)^{1-\alpha}f^{(2)}(s)ds}$$

where symbol Γ **denotes the Euler gamma function.**

Moreover,

A **is** ***n×n*** **dimensional constant matrix with real coefficients,**

admissible controls $u \in U_{ad} = L^2([t_0,t_1],R^p)$ **are unconstrained,**

h>0 **is given constant delay**

$x(t) \in R^n$ **for** $t \in [t_0 - h, t_0]$**,**

$u(t) \in R^p$ **for** $t \in [t_0 - h, t_0]$**,**

$x_{t_0}(t),\ t \in [t_0 - h, t_0]$ **is given continuous initial function,**

$u_{t_0}(t),\ t \in [t_0 - h, t_0]$ **is given initial admissible control**

$B(t,\tau)$ **is $n\times p$ dimensional matrix continuous in t for fixed τ and of bounded variation in τ on *[-h,0]* for each** $t \in [t_0, t_1]$ **and continuous from left in τ on the interval *(-h,0)*.**

integral term in (1) is in the Lebesque-Stieltjes sense [6], [8], [9] with respect to τ,

symbol dB_τ **denotes the Lebesque-Stieltjes integration [6], [8], with respect to the variable τ in the matrix function** $B(t,\tau)$.

initial data $\{x_{t_0}, u_{t_0}\}$ **forms complete state of the fractional delayed system (1) at initial time t_0.**

In order to find the solution of second order fractional differential equation (1) let us use Laplace transform L [11], [14],

$$L[D^\alpha x(t)](s) = s^\alpha L[x(t)](s) - \sum_{k=0}^{k=n-1} x^{(k)}(0) s^{\alpha-1-k}$$

Therefore,

$$X(s) = \left[\frac{s^{\alpha-1}}{s^\alpha I - A - Ce^{-hs}}\right]\varphi(0) + \left[\frac{s^{\alpha-2}}{s^\alpha I - A - Ce^{-hs}}\right]\varphi^{(1)}(0) +$$

$$+ C\left[\frac{e^{-hs}}{s^\alpha I - A - Ce^{-hs}}\right] * \int_{-h}^{0} e^{-s\tau}\,\varphi(\tau)\,d\tau$$

where symbol $*$ dotes convolution.

Hence

$$x(t) = L^{-1}\left[s^{\alpha-1}\left(s^{\alpha}I - A - C^{-hs}\right)^{-1}\right](t)\varphi(0) +$$

$$+ L^{-1}\left[s^{\alpha-1}\left(s^{\alpha}I - A - C^{-hs}\right)^{-1}\right](t)\varphi^{(1)}(0) +$$

Taking into account the above equalities, let us introduce the following matrices

$$X_{\alpha}(t) = L^{-1}\left[s^{\alpha-1}\left(s^{\alpha}I - A - C^{-hs}\right)^{-1}\right](t)$$

$$X_{\alpha,2}(t) = t^{-1}L^{-1}\left[s^{\alpha-2}\left(s^{\alpha}I - A - C^{-hs}\right)^{-1}\right](t)$$

$$X_{\alpha,\alpha}(t) = t^{1-\alpha}\int_{0}^{t}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}X_{\alpha}(s)ds$$

Hence using the above defined matrices solution of the equation (1) with initial condition (2), but without admissible controls is given by the following form

$$x_L\left(t_1, x_{t_1}\right) = X_{\alpha}(t)\varphi(0) + X_{\alpha,2}(t)\varphi^{(1)}(0) +$$

$$+ B\int_{-h}^{0}(t-s-h)^{\alpha-1}X_{\alpha,\alpha}(t-s-h)\varphi(s)ds$$

For first order fractional derivative when $0 < \alpha \leq 1$ the above matrices have the following form

$$\boldsymbol{X_\alpha(t) = L^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha - A - Ce^{-s}}\right](t)}$$

$$\boldsymbol{X_{\alpha,\alpha}(t) = t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} X_\alpha(s)ds}$$

III CONTROLLABILITY CONDITIONS

Since in the paper only relative controllability is considered, then let us recall definition of global relative controllability in a given finite time interval.

Definition 1. The system (1) is said to be globally relatively controllable over time interval $[t_0, t_1]$ **if for each initial complete state** $\{x_{t_0}, u_{t_0}\}$ **of and any final relative state** $x_1 \in R^n$ **there exists an admissible control** $u \in L^2([t_0, t_1], R^p)$ **such that the solution of equation (1) with initial conditions (2) satisfies final condition** $x(t_1) = x_1$**.**

Taking into account the results of section II, solution of equation (1) with admissible controls can be expressed as follows

$$x(t_0, t_1, x_0, u_{t_0}, u) = x_L(t_1, x_{t_0}) + \\ + \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) \left[\int_{-h}^{0} d_\tau B(s,\tau) u(s+\tau) \right] ds \qquad \textbf{(2)}$$

where

$$x_L(t_1, x_{t_0}) = X_\alpha(t) x(t_0) + \\ + \int_{-h}^{0} (t_1 - s - h)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s - h) x_{t_0}(s) ds$$

Now, using unsymmetric Fubini theorem (see e.g. [6] and [8] for more details) and changing order of integration in the last term we have [2], [19], [26]

$$x(t_0,t_1,x_0,u_{t_0},u) = x_L(t_1,x_{t_0}) +$$
$$+\int_{-h}^{0} dB_\tau \left[\int_{t_0}^{t_1} (t_1-s)^{\alpha-1} X_{\alpha,\alpha}(t_1-s) B(s,\tau) u(s)ds \right] =$$
$$= x_L(t_1,x_{t_0}) +$$
$$+\int_{-h}^{0} dB_\tau \left[\int_{\tau}^{t_0} (t_1-s)^{\alpha-1} X_{\alpha,\alpha}(t_1-s) B(s-\tau,\tau) u_{t_0}(s)ds \right] +$$
$$+\int_{-h}^{0} dB_\tau \left[\int_{t_0}^{t_1+\tau} (t_1-s)^{\alpha-1} X_{\alpha,\alpha}(t_1-s) B(s-\tau,\tau) u(s)ds \right] =$$
$$= x_L(t_1,x_{t_0}) +$$
$$+\int_{-h}^{0} dB_\tau \left[\int_{\tau}^{t_0} (t_1-s)^{\alpha-1} X_{\alpha,\alpha}(t_1-s) B(s-\tau,\tau) u_{t_0}(s)ds \right] +$$
$$+\int_{t_0}^{t_1} \left[\int_{-h}^{0} (t_1-s)^{\alpha-1} X_{\alpha,\alpha}(t_1-s) d_\tau B_{t_1}(s-\tau,\tau) \right] u(s)ds \qquad \textbf{(3)}$$

where

$$B_{t_1}(s,\tau) = \begin{cases} B(s,\tau), & s \le t_1, \\ 0, & s > t_1 \end{cases}$$

The first two terms in formula (3) are depended only on given initial complete state $\{x_{t_0}, u_{t_0}\}$, and in fact do not depend on admissible control $u(t),\ t \ge t_0$.Therefore, in order to separate these terms let us denote

$$q(t_1,t_0,x_{t_0},u_{t_0}) = x_L(t_1,x_{t_0}) +$$
$$+\int_{-h}^{0} dB_\tau \left[\int_{t_0+\tau}^{t_0} (t_1-s)^{\alpha-1} X_{\alpha,\alpha}(t_1-s) B(s-\tau,\tau) u_{t_0}(s)ds \right] \qquad \textbf{(4)}$$

Moreover, changing variables in the integral term

$$\left[\int_{-h}^{0} d_{\tau}B(s,\tau)u(s+\tau)\right]$$

and taking into account the form of solution (3) we obtain

$$x(t_0,t_1,x_0,u_{t_0},u)=q(t_1,t_0,x_0,u_{t_0})+$$
$$+\int_{t_0}^{t_1}\left[\int_{-h}^{0}(t_1-s)^{\alpha-1}X_{\alpha,\alpha}(t_1-s)d_{\tau}B_{t_1}(s-\tau,\tau)\right]u(s)ds \qquad \textbf{(5)}$$

Now, let us introduce relative controllability operator $C_{\alpha}(t_1)$ and its adjoint operator $C_{\alpha}^{*}(t_1)$

$$C_{\alpha}(t_1)u=$$
$$=\int_{t_0}^{t_1}(\int_{-h}^{0}t_1-s)^{\alpha-1}X_{\alpha,\alpha}(t_1-s)d_{\tau}B_{t_1}(s-\tau,\tau))u(s)ds \qquad \textbf{(6)}$$

$$C_{\alpha}^{*}(t_1)y=$$
$$=(\int_{-h}^{0}t_1-s)^{\alpha-1}X_{\alpha,\alpha}(t_1-s)d_{\tau}B_{t_1}(s-\tau,\tau))^{*}y \qquad \textbf{(7)}$$

Finally, let us define *n*×*n* dimensional relative controllability matrix

$$W(t_0,t_1) = C_\alpha(t_1)C_\alpha^*(t_1) =$$
$$= \int_{t_0}^{t_1} \Big(\int_{-h}^{0} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) d_\tau B_{t_1}(s-\tau,\tau) \Big) \times \tag{8}$$
$$\times \Big(\int_{-h}^{0} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s)^* d_\tau B_{t_1}(s-\tau,\tau) \Big)^* ds$$

Using relative controllability matrix it is possible to formulate and prove main result of the paper given the following theorem, which presents necessary and sufficient conditions for global relative controllability in a given time interval.

Theorem 1. The following statements are equivalent

(1) **Fractional system (1) is globally relatively controllable over** $t \in [t_0, t_1]$**.**
(2) **Relative controllability linear operator** $C_\alpha : L^2([t_0,t_1], R^p) \to R^n$ **is onto.**
(3) **Adjoint relative controllability operator** $C_\alpha^* : R^n \to L^2([t_0,t_1], R^m)$ **is invertible i.e., it is linear "one to one" operator.**
(4) **The bounded linear operator** $C_\alpha C_\alpha^* : R^n \to R^n$ **is onto and may be realized by** ***nxn*** **nonsingular matrix.**

Proof.
In the proof of Theorem 1 relative controllability linear bounded operator C_α **and its adjoint operator** C_α^* **play the important role. Hence, linear functional analysis theory may be applied to prove theorem. More precisely, we shall use methods and results taken directly from theory of linear bounded operators in Hilbert spaces.**

First of all, let use, that range of the relative controllability operator C_α **is finite dimensional, then operator** C_α **is a bounded linear operator.**

Moreover, as was mentioned before, from the definition 1 and integral formula (8) immediately follows that global relative controllability property is equivalent that relative controllability operator C_α **is surjective operator. Hence, equivalence (1) and (2) follows.**

From the theory of linear operators follows that surjectivity of the operator C_α **implies (see e.g. [8], [31]) that its adjoint**

linear operator

$$C_\alpha^* : R^n \to L^2([t_0, t_1], R^m)$$

is also linear and bounded operator and moreover it is invertible operator, i.e. "one to one" operator.

Hence, equivalence (2) and (3) follows.

Similarly, from theory of linear bounded operators follows, that invertibility of the selfadjoint operator $C_\alpha C_\alpha^*$ means, that exist inverse bounded linear operator $(C_\alpha C_\alpha^*)^{-1}$ and this is equivalent to surjectivity of the operator C_α. Therefore, for relatively controllable fractional system (1), relative controllability matrix

$$W(t_0, t_1) = C_\alpha C_\alpha^* : R^n \to R^n$$

is invertible i.e., it is full rank matrix. Hence, equivalence (4) and (1) follows. This statement completes proof of Theorem 1.

Corollary 2. Fractional system (1) with distributed delay in admissible control is globally relatively controllable on time interval $[t_0, t_1]$ if and only if the relative controllability matrix is nonsingular.

Proof. From global relative controllability definition directly follows, that for relatively controllable fractional system (1) the operator relative controllability operator $C_\alpha(t_1)$ is onto. On the other hand by Theorem 1 this is equivalent, that relative controllability matrix $W(t_0, t_1)$ is nonsingular.

For globally relatively controllable fractional system (1) it is possible to find an admissible control, which transforms given initial complete state $\{x_{t_0}, u_{t_0}\}$ of and any final relative state $x_1 \in R^n$ at time t_1. First of all, let us observe, that since relative controllability matrix $W(t_0, t_1)$ is nonsingular matrix so its inverse $W^{-1}(t_0, t_1)$ is well defined. Therefore, let us define admissible control as follows

$$u^0(t) = C_\alpha^*(t_1)W^{-1}(t_0,t_1)(x_1 - x_L(t_1, x_{t_0})) -$$
$$-\int_{-h}^{0} dB_\tau \left[\int_{t_0+\tau}^{t_0} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) B(s-\tau,\tau) u_{t_0}(s) ds \right] = \qquad \textbf{(9)}$$
$$= C_\alpha^*(t_1)W^{-1}(t_0,t_1)(x_1 - q(t_0,t_1,x_0,u_{t_0}))$$

where complete initial state and the final relative state vector are chosen arbitrarily.

Inserting admissible control $u^0(t)$ **given by equality (9) into solution formula (3) and taking into account equalities (6), (7) and (8) we have**

$$x(t_0,t_1,x_0,u_{t_0},u) = q(t_1,t_0,x_0,u_{t_0}) +$$
$$+\int_{t_0}^{t_1} \left[\int_{-h}^{0} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) d_\tau B_{t_1}(s-\tau,\tau) \right] u^0(s) ds =$$
$$= q(t_1,t_0,x_0,u_{t_0}) +$$
$$+ C_\alpha(t_1) C_\alpha^*(t_1) W^{-1}(t_0,t_1)(x_1 - q(t_1,t_0,x_0,u_{t_0})) = \qquad \textbf{(10)}$$
$$= q(t_1,t_0,x_0,u_{t_0}) +$$
$$+ W(t_0,t_1) W^{-1}(t_0,t_1)(x_1 - q(t_1,t_0,x_0,u_{t_0})) = x_1$$

Thus, the admissible control $u^0(t)$ **transfers initial complete state** $\{x_{t_0}, u_{t_0}\}$ **to final relative state** $x_1 \in R^n$ **at time** t_1**.**

IV CONCLUSIONS

In this paper linear fractional finite-dimensional stationary dynamical control systems with different types of delays in admissible control are considered. More exactly, single constant delay in state variables and distributed delays in admissible controls are discussed. It is generally assumed, that the mathematical model is represented by linear ordinary fractional differential state equations. Using notations, theorems and methods taken directly from functional analysis and linear controllability theory, necessary and sufficient conditions for global relative controllability in a given finite time interval are formulated and proved.

The main result of this paper is to show and to prove, that global relative controllability of fractional control systems with delays both in state variables and in admissible control is equivalent to non-singularity of suitably defined square relative controllability matrix.

Using suitably defined relative controllability matrix for global relatively controllable systems steering admissible control is proposed, which steers the fractional system from given initial complete state to desired final relative state. Moreover, at the beginning of the paper some remarks and comments on the existing in literature controllability results for different types of linear continuous-time and discrete-time fractional dynamical system are also presented.

It should be pointed out, that using different methods of functional analysis, controllability results presented in this paper may be extended in many different ways both for fractional systems and for standard systems [16], [17], [33] and for fractional systems with constrained admissible controls [9], [10], [11], [12], [26], [27] and [28]. First of all, using relative controllability matrix, relative controllability problems for semilinear, or generally nonlinear fractional control systems with different types of delays not only in admissible controls, but also in the state variables recently have been considered in papers [4], [5].

Second possibility is to formulate and prove necessary and sufficient conditions for relative controllability of fractional control systems with different orders of derivatives, applying methods and concepts proposed in paper [12].

The third direction is to consider infinite dimensional control systems applying functional analysis methods and concepts (see monographs [18], [21], and [32]). Since in this case relative state space is infinite dimensional space, then several additional concepts of controllability should be introduced, namely: approximate absolute controllability and exact absolute controllability, approximate relative controllability and exact relative controllability.

In last few years nonlinear or semilinear fractional control systems have been discussed in the literature e. g., in papers [3], and [4]. However, so far, but only rather little attention reports on the global or local relative controllability for delayed systems were published. It follows from the fact, that for nonlinear or semilinear fractional systems we do not know the exact form of the solution for the nonlinear state equation.

Relative controllability conditions for semilinear fractional systems with dominated linear part are discussed in the papers [24], and [25] under the assumption, that linear part is relatively controllable and the nonlinear part satisfy certain inequality.

Generally, in the case of semilinear or nonlinear fractional control systems different techniques are used. The most popular is the fixed-point technique. For example, it is possible to use Banach fixed point theorem, Schauder fixed point theorem, Schaefer fixed point theorem or Darbou fixed point theorem based on measures of noncompactnes in Banach, spaces [6], [8]. It strongly depends on the form of nonlinear part of the fractional state equation.

Minimum energy control problem for fractional systems, similarly as for standard linear systems is strongly connected with different controllability concepts, (see e.g., [16], [20], [24] for more details). First of all, let us observe, that for relatively controllable linear control system there exists generally many different admissible controls transferring given initial state complete state to the desired final relative state. Therefore, we may ask which, of these possible admissible controls are optimal one according to given a priori criterion.

For quadratic criterion and relatively controllable linear fractional systems (1), solution of this problem can be found using relative controllability matrix. Moreover, minimum energy value may be computed in rather simple form. However, it should be mentioned, that this method requires many additional restrictive assumptions (see monographs [18] and [21] and survey papers [16] and [17] for more details) as for example, that state variables and admissible controls are unbounded in whole time interval.

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REFERENCES

[1] R. L. Bagley, and R.A. Calico, “Fractional order state equations for the control of viscoelastically damped structure”, Journal of Guidance, Control and Dynamics, vol. 14, no.2, pp.304-311, 1991.

[2] K. Balachandran,, Y. Zhou and J. Kokila, “Relative controllability of fractional dynamical systems with distributed delays in control”, Computers and Mathematics with Applications (AMCS), vol. 64, no.10, pp.3201-3209, 2012.

[3] K. Balachandran, and J. Kokila, “On the controllability of fractional dynamical systems”, International Journal of Applied Mathematics and Computers Science, vol. 22, no. 3, pp. 523-531, 2012.

[4] K. Balachandran, V. Govindaraj, L. Rdriguez-Germa, and J.J. Trujillo , “Controllability results for nonlinear fractional order dynamical systems”, Journal of Optimization Theory and Applications (JOTA), vol. 156, no. 1, pp. 33-44, 2013.

[5] K. Balachandran, V. Govindaraj, L. Rdriguez-Germa, and J.J. Trujillo, “Controllability of nonlinear higher order fractional dynamical systems, Nonlinear Dynamics, 71(4), 605-612, 2013.

[6] P. Billingsley, “Probability and Measure”, chapter 3 Integration, section 18 Product Measure and Fubini’s Theorem, Wiley, New York, 231-240, 1995.

[7] R.H.Cameron, and W.T. Martin, “An unsymmetric Fubini theorem”, Bulletin of the American Mathematical Society, vol. 47, no. 2, pp. 121-125, 1941.

[8] N. Dunford, and T.J. Schwartz, “Linear Operators”. Part 1. Chapter 8, Operators and Their Applications. Wiley, New York, 1988.

[9] T. Kaczorek, “Fractional positive continuous-time linear systems and their reachability”. International Journal of Applied Mathematical Computation Science (AMCS), vol.18, no.2, pp.223-228, 2008.

[10] T. Kaczorek, “Positive fractional 2D continuous-discrete time linear systems”, Bulletin of the Polish Academy of Sciences, Technical Sciences, vol. 59, no. 4, pp. 575-579, 2011.

[11] T. Kaczorek, “Selected Problems of Fractional Systems Theory”, Springer-Verlag, Berlin, 2012.

[12] T. Kaczorek, “Positive linear systems with different fractional orders”, Bulletin of the Polish Academy of Sciences, Technical Sciences, vol. 58, no. 3, pp. 453-458, 2010.

[13] T. Kaczorek, and K. Rogowski, “Fractional Linear Systems and Electrical Circuits”, Studies in Systems, Decision and Control, vol. 13, Springer, Berlin, 2015.

[14] L. Kexue,, and P. Jigen, “Laplace transform and fractional differential equations”, Applied Mathematic Letters, vol. 24, no. 12, pp. 2019-2023, 2011.

[15] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, “Theory and Applications of Fractional Differential Equations”, Elsevier, Amsterdam, 2006.

[16] J. Klamka, “Relative controllability and minimum energy control of linear systems with distributed delays in control,” IEEE Transactions on Automatic Control, AC-21, no.4, pp.594-595, 1976.

[17] J. Klamka, Relative and absolute controllability of discrete systems with delays in control. International Journal of Control, vol. 26, no. 1, pp. 65-74, 1977.

[18] J. Klamka, “Controllability of Dynamical Systems”. Kluwer Academic Publishers. Dordrecht. The Netherlands. 1991.

[19] J. Klamka, “Controllability of dynamical systems- a survey”, Archives of Control Sciences, vol. 2 (XXXVIII), no. 3-4, pp. 281-307, 1993.

[20] J. Klamka, “Controllability of dynamical systems. A survey”, Bulletin of the Polish Academy of Sciences Technical Sciences, vol. 61, no. 2, pp. 335-342, 2013.

[21] J. Klamka, ”Controllability and Minimum Energy Control”, Monograph in series Studies in Systems, Decision and Control, vol. 162, pp. 1-175, Springer Verlag, Berlin, 2018.

[22] W. Mitkowski, “Approximation of fractional diffusion wave equation”, Acta Mechanica et Automatica, vol. 5, no. 2, pp.65-68, 2011.

[23] W. Mitkowski and P. Skruch, “Fractional-order models of the supercapacitors in the form of RC ladder networks”, Bulletin Polish Academy of Sciences, Technical Sciences, vol. 61, no. 3, pp.581-587, 2013.

[24] R.J. Nirmala, K. Balachandran, L. Rdriguez-Germa, and J.J. Trujillo, “Controllability of nonlinear fractional delay dynamical systems”, Reports on Mathematical Physics, vol. 77, no.1, pp.87-104, 2016.

[25] R.J. Nirmala, and K. Balachandran, “The control of nonlinear implicit fractional delay dynamical systems”, International Journal of Applied Mathematics and Computers Science, (AMCS), vol. 27, no. 3, pp. 501-513, 2017.

[26] B. Sikora, and J. Klamka, “Constrained controllability of fractional linear systems with delays in control”, Systems and Control Letters, vol. 106, no. 1, pp.9-15, 2017.

[27] B. Sikora, and J. Klamka, “Cone-type constrained relative controllability of semilinear fractional systems with delays”, Kybernetika, vol. 53, no. 2, pp. 370-381, 2017.

[28] B. Sikora, and J. Klamka, “New controllability criteria for fractional systems with varying delays”, in Theory and Applications of Non-integer Order Systems. Editors: Babiarz Artur, Czornik Adam, Klamka Jerzy, Niezabitowski Michał. Lecture Notes in Electrical Engineering, vol. 407, Springer Verlag, Berlin, 2017.

[29] M. Sivabalan, and K. Sathiyanathan, “Relative controllability results for

nonlinear higher order fractional delay integrodifferential systems with time varying delay in control", Communications Faculty of Sciences, University of Ankara, series A1, Mathematics and Statistics, vol. 68, no.1, pp.889-906, 2019.

[30] P.J. Torvik, and R. L. Bagley, "On the appearance of the fractional derivative in the behavior of real materials", Journal of Applied Mechanics, vol. 51, no. 2, pp. 294-298, 1984.

[31] G. Venkatesan, and R. George, "Controllability of fractional dynamical systems. A functional analytic approach". Mathematical Control and Related Fields. vol. 7, no. 4, pp. 537-562, 2017.

[32] J.R. Wang, Y. Zhou, and M. Feckan, "Abstract Cauchy problem for fractional differential equations", Nonlinear Dynamics, vol. 71, no. 4, pp. 685-700, 2013.

[33] J. Wei, "The controllability of fractional control systems with control delay", Computers and Mathematics with Applications, vol. 64, no. 10, pp. 3153-3159, 2012.