

Space and time renormalization in phase transition dynamics

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When a system is driven across a quantum critical point at a constant rate, its evolution must become nonadiabatic as the relaxation time τ diverges at the critical point. According to the Kibble-Zurek mechanism (KZM), the emerging post-transition excited state is characterized by a finite correlation length $\hat{\xi}$ set at the time $\hat{t} = \hat{\tau}$ when the critical slowing down makes it impossible for the system to relax to the equilibrium defined by changing parameters. This observation naturally suggests a dynamical scaling similar to renormalization familiar from the equilibrium critical phenomena. We provide evidence for such KZM-inspired spatiotemporal scaling by investigating an exact solution of the transverse field quantum Ising chain in the thermodynamic limit.

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I. INTRODUCTION

The study of the dynamics of second-order phase transitions started in the cosmological setting with the observation by Kibble [1,2] that in the course of the rapid cooling that follows the Big Bang, distinct domains of the nascent Universe will be forced to choose broken symmetry vacua independently. Their incompatibility will typically lead to topological defects that may have observable consequences.

The relativistic causal horizon is no longer a useful constraint in condensed-matter settings, but one can still define a sonic horizon that plays a similar role [3–5]. The usual estimate of the sonic horizon relies on the scaling of the relaxation time and of the healing length, which depend on the dynamical and spatial critical exponents z and ν characteristic for the relevant universality class. The estimate predicts a characteristic time scale $\hat{t} \sim \tau_Q^{z\nu/(1+z\nu)} \sim \hat{\tau}$ and a correlation length (length scale) $\hat{\xi} \sim \tau_Q^{\nu/(1+z\nu)}$, where the quench time τ_Q quantifies the rate of the transition. The correlation length enables prediction of the scaling exponent that governs the number of the generated excitations (e.g., the density of topological defects, when the relevant homotopy group allows for their formation) as a function of τ_Q for a wide range of quench rates.

The Kibble-Zurek mechanism (KZM) has been confirmed by numerical simulations [6–19] and, to a lesser degree and with more caveats, by experiments [20–37] in a variety of settings, with the most recent results in solid-state physics as well as in gaseous Bose-Einstein condensates providing suggestive evidence of KZM scalings [31,32,38–41].

Refinements and extensions of KZM include a phase transition in inhomogeneous systems (see [42] for recent overview) and applications that go beyond topological defect creation (see, e.g., [43–46]). Recent reviews related to the KZ mechanism are also available [47–51].

We consider a zero-temperature quantum phase transition in the transverse-field quantum Ising chain. Despite important differences with respect to thermodynamic phase transitions—where thermal rather than quantum fluctuations act as seeds of symmetry breaking—the KZM can be generalized to quantum phase transitions [52–57]; see, also, [49–51] for reviews. The quantum regime was also addressed in some of the recent experiments [58–62].

In this paper, we propose what can be considered a generalization and extension of the predictive power of KZM: In the adiabatic limit, when $\tau_Q \rightarrow \infty$, both \hat{t} and $\hat{\xi}$ diverge. Hence, one can expect that they should be the only relevant time and length scales in the low-frequency and long-wavelength regime. This in turn suggests a dynamical scaling hypothesis, similar to the one that underlies the renormalization paradigm that is so useful for the equilibrium phase transitions, that during the quench all physical observables depend on time t through the rescaled time t/\hat{t} and on a distance x through the rescaled distance $x/\hat{\xi}$. Though the basic ingredients of the hypothesis were present in the KZM from the beginning (see, e.g., the discussion of the rescaling of the Gross-Pitaevskii equation in [5], as well as [43,63–65]), its fully fledged form, taking into account the scaling dimension, was articulated first in [66] for the correlation function of the ferromagnetic order parameter in the quantum Ising chain. The idea was developed further in [67].

Our aim here is a comprehensive study of this space-time renormalizationlike scaling in the exactly solvable Ising chain. We begin with a general discussion of the quantum KZM in Sec. II. It is followed by the statement of the KZM scaling hypothesis in Sec. III. In Sec. IV, we discuss the sonic horizon. The Ising model is solved in Secs. V and VA by mapping to a set of independent Landau-Zener (LZ) systems. The scaling in the LZ context is identified in Sec. VB. Then the same scaling is found in quadratic fermionic correlators (Sec. VC), energy and quasiparticle density (Sec. VD), spin-spin correlators (Sec. VE), mutual information (Sec. VF), quantum discord (Sec. VG), entropy of entanglement (Sec. VH), and entanglement gap (Sec. VI). We conclude in Sec. VI.

II. QUANTUM KIBBLE-ZUREK MECHANISM

A distance from a quantum critical point can be measured with a dimensionless parameter ϵ . The ground state of the Hamiltonian $H(\epsilon)$ changes character (e.g., breaks a symmetry) when $\epsilon = 0$. Thus, ϵ plays a role analogous to the relative temperature in thermodynamic phase transitions.

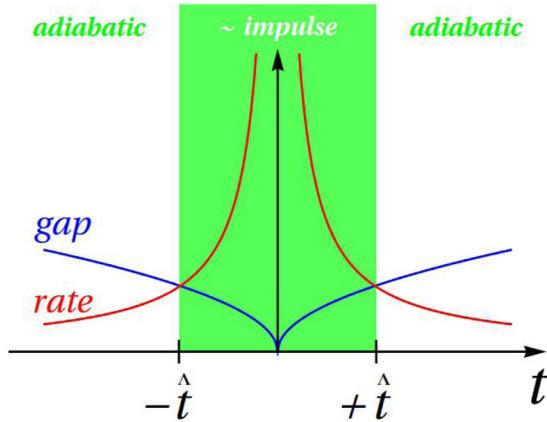


FIG. 1. Schematic illustration of the quantum Kibble-Zurek mechanism. The Hamiltonian is driven by a linear quench (3) across the critical point at $t = 0$. The energy gap (2) closes like $|\epsilon|^{z\nu}$, where z and ν are universal exponents, and the transition rate (4) diverges at the critical point; hence the evolution cannot be adiabatic between $-\hat{t}$ and $+\hat{t}$. This figure shows the gap for $z = 1$ and $\nu = 1/2$, i.e., the mean-field scalings. In the quantum Ising model, $z = \nu = 1$ and the gap closes linearly as $t \rightarrow 0$.

The correlation length in its ground state diverges as

$$\xi \sim |\epsilon|^{-\nu}, \quad (1)$$

and the relevant gap closes,

$$\Delta \sim |\epsilon|^{z\nu}; \quad (2)$$

see Fig. 1. The system, initially prepared in its ground state, is driven across the critical point by a linear quench,

$$\epsilon(t) = \frac{t}{\tau_Q}, \quad (3)$$

with a quench time τ_Q . Nonlinear “protocols” can also be considered [67,68], but we shall not deal with them here.

The evolution sufficiently far from the critical point is initially adiabatic. However, the rate of change of epsilon,

$$\left| \frac{\dot{\epsilon}}{\epsilon} \right| = \frac{1}{|t|}, \quad (4)$$

diverges at the gapless critical point. Therefore, evolution (e.g., of the order parameter) cannot be adiabatic in its neighborhood between $-\hat{t}$ and $+\hat{t}$; see Fig. 1. Here, \hat{t} is the time when the gap (2) equals the rate (4), so that

$$\hat{t} \sim \tau_Q^{z\nu/(1+z\nu)} \sim \hat{t}. \quad (5)$$

Just before the adiabatic-to-nonadiabatic crossover at $-\hat{t}$, the state of the system is still approximately the adiabatic ground state at $\epsilon = -\hat{\epsilon}$, where

$$\hat{\epsilon} = \frac{\hat{t}}{\tau_Q} \simeq \tau_Q^{-1/(1+z\nu)}, \quad (6)$$

with a correlation length

$$\hat{\xi} \sim \hat{\epsilon}^{-\nu} \sim \tau_Q^{\nu/(1+z\nu)}. \quad (7)$$

In a zeroth-order impulse approximation (which is the “cariature” of the KZM often found in papers), this state “freezes

out” at $-\hat{t}$ and literally does not change until \hat{t} . At \hat{t} , the frozen state is no longer the ground state but an excited state with a correlation length $\hat{\xi}$. It is the initial state for the adiabatic process that follows after \hat{t} .

There are cases where this oversimplified view suffices [53]. Moreover, as we shall see below, it predicts the same scalings for $\hat{\xi}$ as the original derivation [3,5] based on the size of the sonic horizon.

III. SPACE-TIME RENORMALIZATION SCALING HYPOTHESIS

No matter how accurate is the impulse approximation or the above “freeze-out scenario,” the scaling argument establishes $\hat{\xi}$ and \hat{t} , interrelated via

$$\hat{t} \sim \hat{\xi}^z, \quad (8)$$

as the relevant scales of length and time. What is more, in the adiabatic limit when $\tau_Q \rightarrow \infty$, both scales diverge, becoming the unique scales in the long-wavelength and low-frequency limit. Like in the static critical phenomena, this uniqueness implies a scaling hypothesis:

$$\langle \psi(t) | O(x) | \psi(t) \rangle = \hat{\xi}^{-\Delta_O} F_O(t/\hat{t}, x/\hat{\xi}). \quad (9)$$

Here, $|\psi(t)\rangle$ is the state during the quench, $O(x)$ is an operator depending on a distance x , F_O is its scaling function, and Δ_O is its scaling dimension. This hypothesis is analogous to the static one in the ground state $|\psi_{GS}\rangle$,

$$\langle \psi_{GS} | O(x) | \psi_{GS} \rangle = \xi^{-\Delta_O^{GS}} F_O^{(GS)}(x/\xi), \quad (10)$$

where ξ is a diverging correlation length near a quantum critical point.

The diverging scales, $\hat{\xi}$ and \hat{t} , become the unique scales in a coarse-grained description at large distances and long times, but the scaling hypothesis is not warranted to hold at short microscopic distances of a few lattice sites, where microscopic scales remain relevant. This is the same as in the static critical phenomena.

The analogy to the static case is nearly an identity near $t/\hat{t} = -1$, where $|\psi(t)\rangle = |\psi_{GS}\rangle$ and $\hat{\xi} = \xi$. Consequently,

$$F_O(-1, x/\hat{\xi}) = F_O^{(GS)}(x/\xi), \quad (11)$$

$$\Delta_O = \Delta_O^{(GS)}. \quad (12)$$

The dynamical dimension is the same as the static one. Exploiting further the adiabaticity before $t/\hat{t} = -1$, the adiabatic scaling function is well approximated by

$$\begin{aligned} F_O(t/\hat{t} < -1, x/\hat{\xi}) &= (\xi/\hat{\xi})^{-\Delta_O} F_O^{(GS)}\left(\frac{x/\hat{\xi}}{\xi/\hat{\xi}}\right) \\ &= (t/\hat{t})^{\nu\Delta_O} F_O^{(GS)}\left[\frac{x/\hat{\xi}}{(t/\hat{t})^{-\nu}}\right]. \end{aligned} \quad (13)$$

Here, ξ is the correlation length in the adiabatic ground state before $-\hat{t}$. It depends on time like $\xi/\hat{\xi} = (\epsilon/\hat{\epsilon})^{-\nu} = (t/\hat{t})^{-\nu}$. What is more, in the impulse approximation, the nonadiabatic scaling function should not depend on the rescaled

time,

$$\begin{aligned} F_O(-1 < t/\hat{t} < 1, x/\hat{\xi}) &= F_O(-1, x/\hat{\xi}) \\ &= F_O^{(\text{GS})}(x/\hat{\xi}). \end{aligned} \quad (14)$$

If accurate, the dynamical function would be completely expressible by the static one.

IV. QUASIPARTICLES AND SONIC HORIZON

However, the reality turns out to be more interesting. In the following, we will see that all scaling functions do depend on t/\hat{t} during the nonadiabatic stage. For instance, in Fig. 2, we show the ferromagnetic correlation function in the quantum Ising chain. Near $t/\hat{t} = 0$, its range grows almost as the size of the “sound cone”—with twice the speed of quasiparticles at the critical point. Between $t/\hat{t} = -1$ and $t/\hat{t} = 1$, it has enough time to increase several times. The quench excites entangled pairs of quasiparticles with opposite quasimomenta that spread correlations across the system [69]. This sonic horizon effect is, in a sense, at odds with the simplistic narrative of the impulse approximation. Indeed, as the correlation range grows with time, it appears to undermine the significance of $\hat{\xi}$ as a preferred scale of length. Nonetheless, in the following, we will see that the KZM scaling holds with $\hat{\xi}$ as the relevant length.

In order to relate scaling deduced from the “freeze-out” picture implied by the impulse approximation (where the evolution pauses in the interval $[-\hat{t}, \hat{t}]$, and the scale $\hat{\xi}$ is “inherited” from the frozen-out pretransition fluctuations) and the view based on causality and sonic horizon, we focus on a quench-induced evolution in the near-critical regime. After $-\hat{t}$, the state must depart from the adiabatic ground state as otherwise its correlation length would diverge at the critical point, since correlations cannot spread infinitely fast. Respecting this speed limit, after the freeze out at $-\hat{t}$, the range of correlations continues to grow, but with a finite speed set by

$$\hat{v} \simeq \hat{\xi}/\hat{t}, \quad (15)$$

given by a combination of the relevant scales that defines the speed of the relevant sound. Indeed, the nonadiabatic evolution excites low-frequency quasiparticles with quasimomenta up to

$$\hat{k} \simeq \hat{\xi}^{-1}. \quad (16)$$

For a quasiparticle dispersion $\propto k^z$ at the critical point, the maximal velocity of the excitations is

$$\hat{v} \sim \hat{k}^{z-1} = \frac{\hat{k}^z}{\hat{k}} \simeq \frac{\hat{\xi}}{\hat{t}} \sim \tau_Q^{-(z-1)v/(1+zv)}. \quad (17)$$

With twice this velocity, the correlation length can grow from the initial $\hat{\xi}$ near $-\hat{t}$ to a final $\hat{\xi} + (2\hat{t})(2\hat{v}) = 5\hat{\xi}$ near \hat{t} . The final length, even though multiplied by factor of ~ 5 , is still proportional to the original $\hat{\xi}$.

A few remarks are in order before we begin to illustrate this discussion with the example of the Ising chain. We first note that even though the impulse approximation is not accurate in general, occasionally it yields remarkably accurate, or even exact, results [70]. The correlation range of $\sim 5\hat{\xi}$ may help explain some of the discrepancy between simple estimates of defect density and numerical simulations (where it was

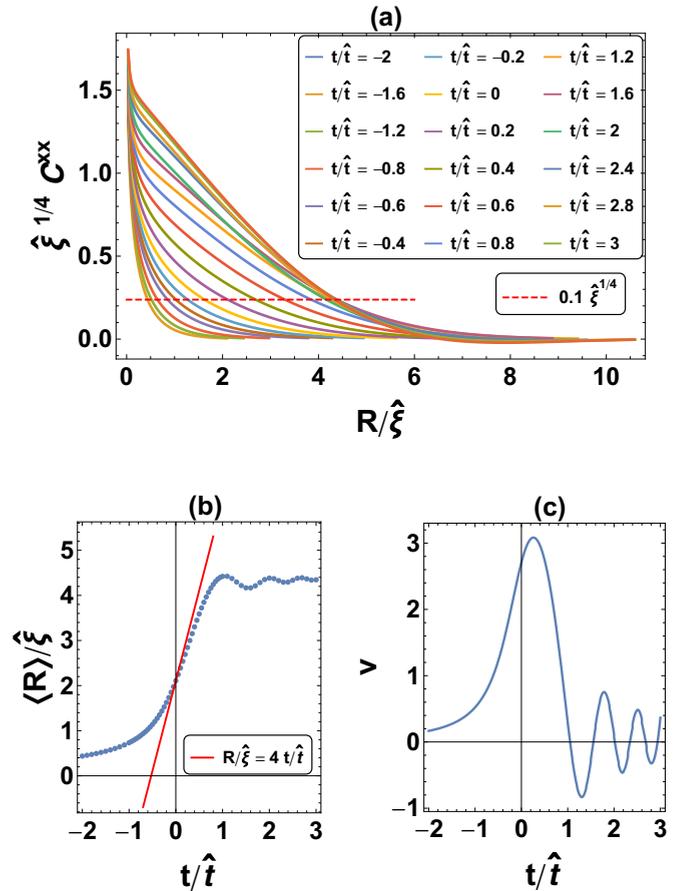


FIG. 2. (a) The ferromagnetic correlation function in the quantum Ising chain (18), $C_R^{xx} = \langle \sigma_n^x \sigma_{n+R}^x \rangle - \langle \sigma_n^x \rangle \langle \sigma_{n+R}^x \rangle$, as a function of the rescaled distance $R/\hat{\xi}$ at different rescaled times $t/\hat{t} = -2, \dots, 3$. Here, σ_n^x is the x component of a spin 1/2 at the lattice site n and $\langle \sigma_n^x \rangle$ is the ferromagnetic magnetization (order parameter). The quench time is $\tau_Q = 1024$. (b) A correlation range $\langle R \rangle$ as a function of time. We define $\langle R \rangle$ as the distance R where the correlator C_R^{xx} falls below 0.1. This cutoff is somewhat arbitrary. The behavior of $\langle R \rangle$ is similar for other cutoffs of this order. It is selected to provide the size of domains that choose the same broken symmetry state (rather than as an estimate of the correlation length, which is defined for much smaller values of C_R^{xx}). The range $\langle R \rangle$ increases several times between $-\hat{t}$ and \hat{t} before it freezes after \hat{t} . Similar plots to (a) and (b) were made in Ref. [66]. (c) The velocity $v = d\langle R/\hat{\xi} \rangle/d(t/\hat{t})$ —the derivative of the plot in (b)—as a function of the rescaled time. Since in the Ising chain $z = 1$ and $\hat{\xi} = \hat{t}$, it is also the velocity in the standard units, $v = d\langle R \rangle/dt$. Near $t/\hat{t} \approx 0$, the correlation is spreading with nearly twice the speed of quasiparticles, $c = 2$, at the critical point, suggesting that “sound cones” (in analogy with “light cones”) are responsible for the size of the domains.

noted that defects are separated by distances of several $\hat{\xi}$ (see, e.g., [6,7,9]). Last, but not least, we also note that the behavior of the speed of sound in the near-critical regime is controlled by the dynamical critical exponent z . In the quantum Ising chain $z = 1$, which means that the speed of sound is constant with respect to the quench time. We can, however, envisage situations where propagation of quasiparticles is impeded (e.g., by damping or conservation laws). That would complicate the

sonic horizon scenario and could even make the “freeze-out paradigm” an accurate approximation.

V. QUANTUM ISING CHAIN

We test the KZM scaling in the quantum Ising chain,

$$H = - \sum_{n=1}^N (g\sigma_n^z + \sigma_n^x \sigma_{n+1}^x), \quad (18)$$

with periodic boundary conditions. For $N \rightarrow \infty$, it has two critical points at $g_c = \pm 1$ between a ferromagnetic phase when $|g| < 1$ and two paramagnetic phases when $|g| > 1$. We assume $g > 0$ for definiteness.

A linear quench runs from $t = -\infty$ and across the critical point when $t = 0$:

$$g(t) = 1 - \frac{t}{\tau_Q} = 1 - \epsilon(t). \quad (19)$$

The critical exponents are $z = \nu = 1$. The KZM yields the temporal and spatial scales,

$$\hat{t} \simeq \sqrt{\tau_Q}, \quad \hat{\xi} \simeq \sqrt{\tau_Q}. \quad (20)$$

In the following exact solutions, we will use definitions $\hat{t} \equiv \sqrt{\tau_Q}$ and $\hat{\xi} \equiv \sqrt{\tau_Q}$.

A. From spins to Landau-Zener model

Here we assume that N is even for convenience. Following the Jordan-Wigner transformation,

$$\sigma_n^x = -(c_n + c_n^\dagger) \prod_{m<n} (1 - 2c_m^\dagger c_m), \quad (21)$$

$$\sigma_n^y = i(c_n - c_n^\dagger) \prod_{m<n} (1 - 2c_m^\dagger c_m), \quad (22)$$

$$\sigma_n^z = 1 - 2c_n^\dagger c_n, \quad (23)$$

we introduce fermionic operators c_n that satisfy $\{c_m, c_n^\dagger\} = \delta_{mn}$ and $\{c_m, c_n\} = \{c_m^\dagger, c_n^\dagger\} = 0$. The Hamiltonian (18) becomes

$$H = P^+ H^+ P^+ + P^- H^- P^-. \quad (24)$$

Above $P^\pm = \frac{1}{2}[1 \pm P]$ are projectors on subspaces with even (+) and odd (−) parity,

$$P = \prod_{n=1}^N \sigma_n^z = \prod_{n=1}^N (1 - 2c_n^\dagger c_n), \quad (25)$$

and

$$H^\pm = \sum_{n=1}^N \left(g c_n^\dagger c_n - c_n^\dagger c_{n+1} - c_{n+1} c_n - \frac{g}{2} \right) + \text{H.c.} \quad (26)$$

are the corresponding reduced Hamiltonians. The c_n 's in H^- satisfy periodic boundary condition $c_{N+1} = c_1$, but the c_n 's in H^+ are antiperiodic: $c_{N+1} = -c_1$.

The initial ground state at $g \rightarrow \infty$ has even parity; hence we can focus on the even subspace. H^+ is diagonalized by a Fourier transform followed by a Bogoliubov transformation.

The antiperiodic Fourier transform is

$$c_n = \frac{e^{-i\pi/4}}{\sqrt{N}} \sum_k c_k e^{ikn}, \quad (27)$$

where the pseudomomentum takes half-integer values,

$$k = \pm \frac{1}{2} \frac{2\pi}{N}, \dots, \pm \frac{N-1}{2} \frac{2\pi}{N}. \quad (28)$$

The Hamiltonian (26) becomes

$$H^+ = \sum_k [2(g - \cos k) c_k^\dagger c_k + \sin k (c_k^\dagger c_{-k}^\dagger + c_{-k} c_k) - g]. \quad (29)$$

Its diagonalization is completed by a Bogoliubov transformation $c_k = U_k \gamma_k + V_k^* \gamma_{-k}^\dagger$, provided that Bogoliubov modes (U_k, V_k) are eigenstates of the stationary Bogoliubov–de Gennes equations

$$\omega_k \begin{pmatrix} U_k \\ V_k \end{pmatrix} = 2[\sigma^z(g - \cos k) + \sigma^x \sin k] \begin{pmatrix} U_k \\ V_k \end{pmatrix}, \quad (30)$$

with a positive eigenfrequency

$$\omega_k = 2\sqrt{(g - \cos k)^2 + \sin^2 k}. \quad (31)$$

The corresponding normalized eigenstate (U_k, V_k) defines a quasiparticle operator, $\gamma_k = U_k c_k + V_k^* c_{-k}^\dagger$, bringing the Hamiltonian to the diagonal form $H^+ = E_0^+ + \sum_k \omega_k \gamma_k^\dagger \gamma_k$. Thanks to the projection $P^+ H^+ P^+$ in Eq. (24), only states with even numbers of quasiparticles belong to the spectrum of H^+ —in a periodic chain, kinks must be created in pairs.

The initial ground state at $g \rightarrow \infty$ is a Bogoliubov vacuum $|0\rangle$ annihilated by all γ_k . As $g(t)$ is ramped down, the state gets excited from the instantaneous ground state, but in the Heisenberg picture it remains the initial vacuum. Instead, the fermionic operators are time dependent,

$$c_k = u_k(t) \gamma_k + v_k^*(t) \gamma_{-k}^\dagger, \quad (32)$$

with the initial condition $(u_k, v_k) = (1, 0)$. They satisfy Heisenberg equations $i \frac{d}{dt} c_k = [c_k, H^+]$ equivalent to the time-dependent Bogoliubov–de Gennes equations (30):

$$i \frac{d}{dt} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = 2\{\sigma^z[g(t) - \cos k] + \sigma^x \sin k\} \begin{pmatrix} u_k \\ v_k \end{pmatrix}. \quad (33)$$

A new time variable for $k > 0$,

$$t' = 4\tau_Q \sin k \left(-1 + \frac{t}{\tau_Q} + \cos k \right), \quad (34)$$

brings Eqs. (33) to the canonical LZ form,

$$i \frac{d}{dt'} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \frac{1}{2} \left[-\frac{t'}{\tau_k} \sigma^z + \sigma^x \right] \begin{pmatrix} u_k \\ v_k \end{pmatrix}, \quad (35)$$

with a transition time $\tau_k = 4\tau_Q \sin^2 k$. The solution of Eqs. (33) is

$$u_k = e^{-\frac{\pi}{16} \tau_k} D_{\frac{1}{4} i \tau_k}(z) e^{i\pi/4}, \quad (36)$$

$$v_k = \frac{1}{2} e^{-\frac{\pi}{16} \tau_k} D_{-1 + \frac{1}{4} i \tau_k}(z) \sqrt{\tau_k}. \quad (37)$$

Here, $D_m(z)$ is the Weber function with an argument,

$$z = e^{3\pi i/4} \frac{t'}{\sqrt{\tau_Q}}. \quad (38)$$

The scaling is not apparent in this exact formula.

B. Scaling in Landau-Zener model

Only small quasimomenta up to

$$\hat{k} = 1/\sqrt{\tau_Q} \quad (39)$$

get excited. For $k \ll 1$, we can approximate

$$\begin{aligned} u_k &= e^{-\frac{1}{4}\pi q^2} D_{iq^2}(z) e^{i\pi/4}, \\ v_k &= e^{-\frac{1}{4}\pi q^2} D_{-1+iq^2}(z) q, \\ z &= 2e^{3\pi i/4} \left(\frac{t}{\hat{t}} - \frac{q^2}{2\sqrt{\tau_Q}} \right), \end{aligned} \quad (40)$$

where $q = k/\hat{k}$ is a rescaled quasimomentum.

Only q up to $q \approx 1$ get excited. For them, when τ_Q is large enough, we can further approximate $z \approx 2e^{3\pi i/4}(t/\hat{t})$ (see Fig. 3) and obtain

$$\begin{aligned} u_k &= e^{-\frac{1}{4}\pi q^2} D_{iq^2} \left(2e^{3\pi i/4} \frac{t}{\hat{t}} \right) e^{i\pi/4}, \\ v_k &= e^{-\frac{1}{4}\pi q^2} D_{-1+iq^2} \left(2e^{3\pi i/4} \frac{t}{\hat{t}} \right) q. \end{aligned} \quad (41)$$

As required by the space-time scaling, these nonadiabatic modes depend on the rescaled t/\hat{t} and q only.

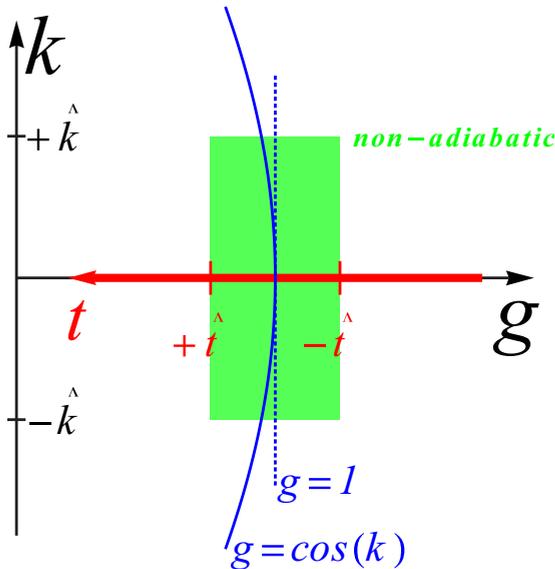


FIG. 3. The Landau-Zener modes (33) with different k pass through their anticrossings at different $g_{AC}(k) = \cos(k)$. In the adiabatic limit, the size of the nonadiabatic regime $\hat{k} \simeq 1/\sqrt{\tau_Q}$ becomes small, $1 - g_{AC}(\hat{k}) \simeq 1/\tau_Q$ becomes much less than $1 - g(\hat{t}) \simeq 1/\sqrt{\tau_Q}$, and we can safely approximate $g_{AC}(k) \approx 1$, as if all of the anticrossings took place simultaneously at the critical $g = 1$. This is the essence of the approximation between Eqs. (40) and (41).

C. Scaling in fermionic correlators

The state during the quench is fully determined by time-dependent quadratic correlators. In the thermodynamic limit $N \rightarrow \infty$, they are given by integrals:

$$\alpha_R(t) \equiv \langle c_R c_0^\dagger \rangle = \frac{1}{\pi} \int_0^\pi dk |u_k|^2 \cos(kR), \quad (42)$$

$$\beta_R(t) \equiv \langle c_R c_0 \rangle = \frac{1}{\pi} \int_0^\pi dk u_k v_k^* \sin(kR). \quad (43)$$

The integrals extend into the adiabatic regime, where the scaling form (41) is no longer applicable. Instead, the modes u_k, v_k can be approximated (up to an irrelevant dynamical phase) by the adiabatic eigenmodes U_k, V_k at $g = 1 - (t/\hat{t})/\sqrt{\tau_Q}$.

In order to demonstrate the scaling of α_R , it is convenient to rearrange it first as

$$\alpha_R = \alpha_R^{(KZ)} + \alpha_R^{(GS)} + \alpha_R^{(cr)}, \quad (44)$$

where

$$\alpha_R^{(KZ)} = \frac{1}{\pi} \int_0^\pi dk (|u_k|^2 - |U_k|^2) \cos(kR), \quad (45)$$

$$\alpha_R^{(GS)} = \frac{1}{\pi} \int_0^\pi dk (|U_k|^2 - |U_k^{CP}|^2) \cos(kR), \quad (46)$$

$$\alpha_R^{(CP)} = \frac{1}{\pi} \int_0^\pi dk |U_k^{CP}|^2 \cos(kR). \quad (47)$$

Here, U_k and U_k^{CP} are the adiabatic eigenmodes at $g = 1 - (t/\hat{t})/\sqrt{\tau_Q}$ and the critical $g = 1$, respectively.

Since the correlation length in the ground state at $g = 1 - (t/\hat{t})/\sqrt{\tau_Q}$ is $\xi \simeq \sqrt{\tau_Q}/(t/\hat{t})$, in Eq. (46) the integrand is nonzero up to $\hat{k} \simeq \xi^{-1}$. Consequently, given that $\xi \simeq \hat{\xi}/(t/\hat{t})$, a change of the integration variable $k \rightarrow k\hat{\xi}$ is enough to show that

$$\alpha_R^{(GS)} = \hat{\xi}^{-1} F_\alpha^{(GS)}(t/\hat{t}, R/\hat{\xi}). \quad (48)$$

Here, F_α is a scaling function.

In a similar way, in Eq. (45) the integrand is nonzero in the nonadiabatic regime up to \hat{k} . In this regime, u_k has the scaling form (41) and U_k has a characteristic quasimomentum scale $\simeq \xi^{-1} \sim \hat{\xi}^{-1}$. Consequently, the same change of the integration variable shows again that

$$\alpha_R^{(KZ)} = \hat{\xi}^{-1} F_\alpha^{(KZ)}(t/\hat{t}, R/\hat{\xi}) \quad (49)$$

for large enough τ_Q .

Finally, Eq. (47) is the ground-state correlator at the critical point,

$$\alpha_R^{(CP)} = \frac{-\frac{1}{4\pi}}{R^2 - \frac{1}{4}} = \hat{\xi}^{-2} \frac{-\frac{1}{4\pi}}{\left(\frac{R}{\hat{\xi}}\right)^2 - \frac{1}{4\hat{\xi}^2}} \approx \hat{\xi}^{-2} \frac{-\frac{1}{4\pi}}{(R/\hat{\xi})^2}. \quad (50)$$

It has a scaling form, but its scaling dimension -2 is twice the -1 in Eqs. (48) and (49). For slow enough quenches, $\alpha_R^{(CP)}$ becomes negligible as compared to the other two terms.

Collecting together Eqs. (48)–(50) and (44), we can conclude with a dynamical scaling law,

$$\alpha_R = \hat{\xi}^{-1} F_\alpha(t/\hat{t}, R/\hat{\xi}), \quad (51)$$

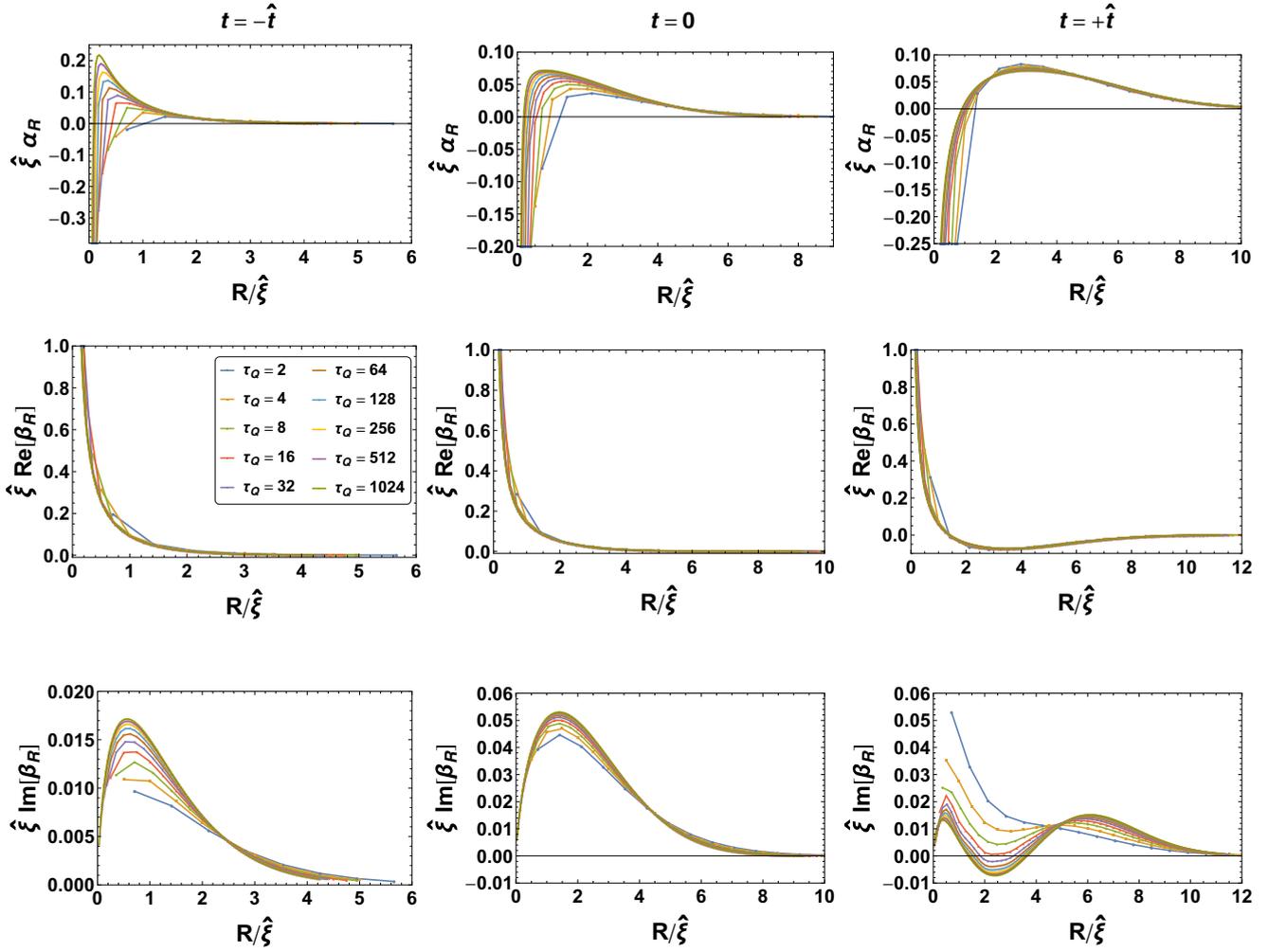


FIG. 4. Quadratic fermionic correlators in Eqs. (42) and (43). The first row shows the real $\alpha_R = \langle c_R c_0^\dagger \rangle$. The second and third rows show the real and imaginary parts of $\beta_R = \langle c_R c_0 \rangle$, respectively. The left, middle, and right columns show the correlators before ($t/\hat{t} = -1$), at ($t/\hat{t} = 0$), and after ($t/\hat{t} = +1$) the critical point, respectively. Different colors of the plots correspond to different quench times τ_Q . All plots are rescaled: they are presented as a function of the rescaled distance $R/\hat{\xi} = R/\sqrt{\tau_Q}$ and the correlators are multiplied with $\hat{\xi} = \sqrt{\tau_Q}$. At large distance, $R \gg 1$, the plots in all nine panels collapse asymptotically with increasing τ_Q , demonstrating the scaling hypotheses (51) and (53) for large enough τ_Q . The collapse does not happen at short distance, $R \approx 1$, where the microscopic lattice constant remains a relevant scale of length and the scaling hypothesis is not expected to hold. The collapsed plots for large τ_Q are the scaling functions $F_\alpha(t/\hat{t}, R/\hat{\xi})$ and $F_\beta(t/\hat{t}, R/\hat{\xi})$.

valid for large enough τ_Q . In Fig. 4, we show rescaled plots supporting this conclusion for large distances $R \gg 1$ and the quench time τ_Q where the scaling hypothesis is expected to hold. The plots were obtained by numerical integration in Eq. (42); see the Appendix.

The argument for β_R is similar except that in the critical ground state, the scaling dimension is -1 :

$$\beta_R^{(\text{CP})} = \frac{1}{R - \frac{1}{4R}} \approx \hat{\xi}^{-1} \frac{1}{(R/\hat{\xi})}. \quad (52)$$

This difference does not alter the overall scaling,

$$\beta_R = \hat{\xi}^{-1} F_\beta(t/\hat{t}, R/\hat{\xi}), \quad (53)$$

with the same dimension. Figure 4 supports this conclusion for large distances $R \gg 1$ and the quench time τ_Q where the scaling hypothesis is expected to hold.

The quadratic correlators completely determine the Bogoliubov vacuum state. They satisfy the KZM scaling. Therefore, it is tantalizing to take the scaling for granted for any operator $O(x)$ in this state. However, as the quadratic correlators satisfy the scaling only asymptotically for slow enough τ_Q , we cannot assume that their convergence with τ_Q , or collapse in Fig. 4, is fast enough to warrant similar collapse for any operator $O(x)$. Therefore, in the following, we study the most interesting observables case by case.

D. Scaling in energy and number of excitations

To begin with operators that do not depend on any distance x , we consider the density of quasiparticle excitations,

$$\frac{n}{N} = \frac{1}{\pi} \int_0^\pi dk p_k, \quad (54)$$

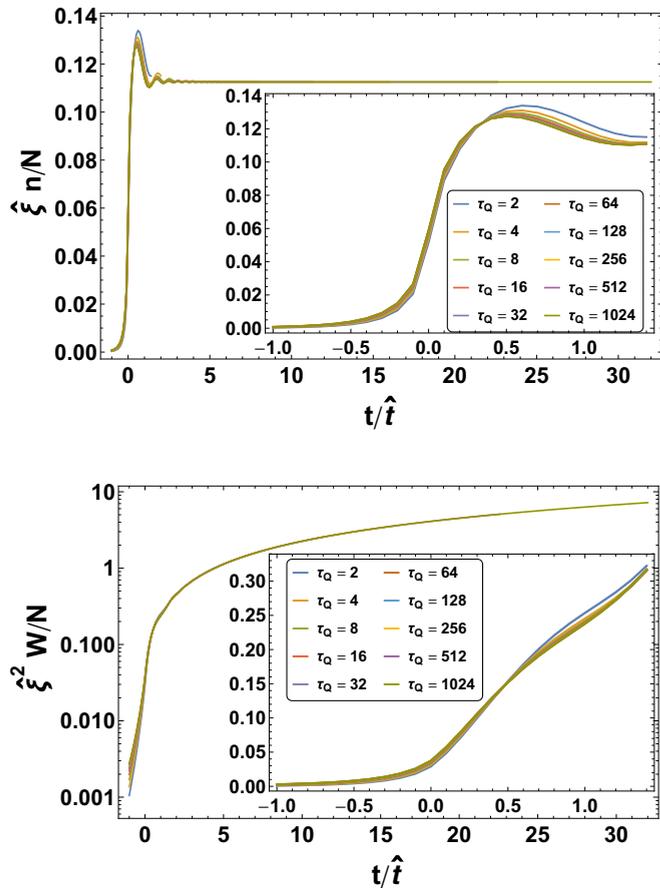


FIG. 5. Top: the density of quasiparticle excitations in Eq. (54) as a function of the rescaled time t/\hat{t} . Different colors of the plots correspond to different quench times τ_Q . Each plot reaches up to $t/\hat{t} = \sqrt{\tau_Q}$, corresponding to zero transverse field $g = 0$. The inset is a focus on the nonadiabatic stage $t/\hat{t} = -1 \dots 1$. Bottom: the density of excitation energy in Eq. (55) as a function of t/\hat{t} . In both panels, the plots collapse asymptotically with increasing τ_Q , demonstrating the scaling hypotheses (57) for large enough τ_Q . The collapsed plots for large τ_Q are the scaling functions $F_n(t/\hat{t})$ and $F_W(t/\hat{t})$ in Eq. (57). The same quantities with a different rescaling of time can be seen in Fig. 6.

and excitation energy,

$$\frac{W}{N} = \frac{1}{\pi} \int_0^\pi dk p_k 2\omega_k, \quad (55)$$

both in the thermodynamic limit $N \rightarrow \infty$. Here, ω_k is the instantaneous quasiparticle dispersion (31) and p_k is the excitation probability for a pair of quasiparticles with quasimomenta $(k, -k)$:

$$p_k = \left| (-V_k, U_k) \begin{pmatrix} u_k \\ v_k \end{pmatrix} \right|^2. \quad (56)$$

Since p_k is nonzero in the nonadiabatic regime only up to \hat{k} and, furthermore, $\omega_k \sim k$ in this regime, a change of the integration variable from k to $k\hat{\xi}$ leads to the scaling forms

$$\hat{\xi}^1 \frac{n}{N} = F_n(t/\hat{t}), \quad \hat{\xi}^2 \frac{W}{N} = F_W(t/\hat{t}). \quad (57)$$

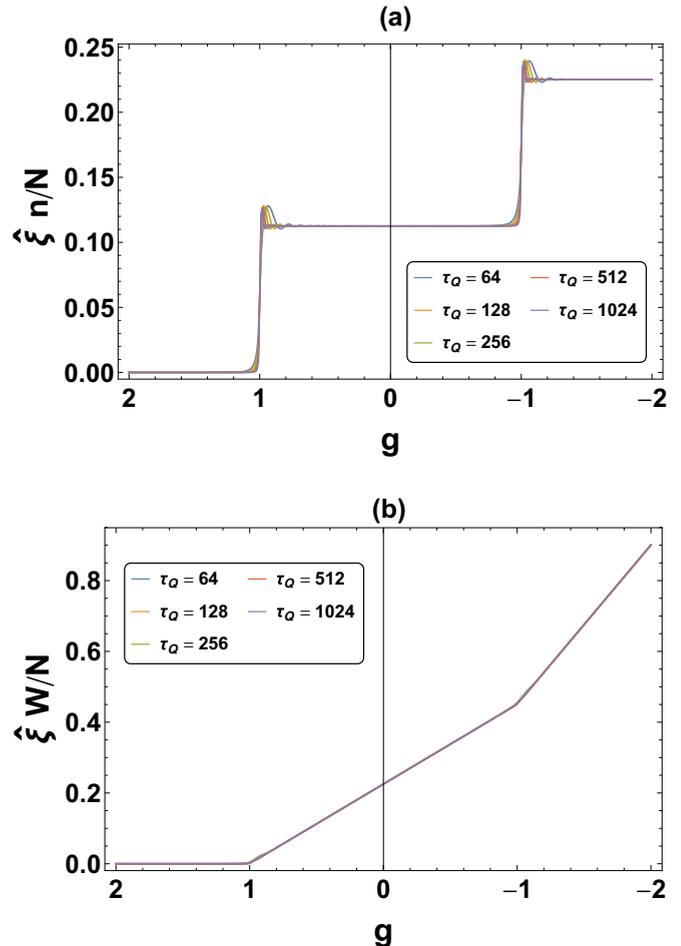


FIG. 6. (a) The density of quasiparticle excitations as a function of g . Quasiparticles are excited at the two critical points, $g = \pm 1$. For any fixed g , except near $g = \pm 1$, their density scales as $n/N \sim \hat{\xi}^{-1}$ (i.e., the typical distance between the kinks is set by $\hat{\xi}$). (b) The density of excitation energy as a function of g . For any fixed g , it scales like $W/N \sim \hat{\xi}^{-1}$, except near the gapless critical points $g = \pm 1$. Apart from the immediate vicinity of the critical points, the evolution is adiabatic. That is, the increase of energy is caused by the increase in the separations between the occupied energy levels. Thus, when the Ising chain is still in its ground state [leftmost segment in (b)], $W = 0$, but after the excitation caused by the passage through the critical point at $g = 1$, the slope increases and remains constant until $g = -1$. Additional excitations double the slope in the rightmost segments of (b). Indeed, these slopes are set by the quasiparticle densities seen in (a).

The last form is consistent with the prediction of Ref. [50] for gapless systems. The collapsing plots in Fig. 5, similar to the plots in Ref. [63], demonstrate this scaling. Interestingly, the work density collapses well beyond $t/\hat{t} = 1$, even though the gapless $\omega_k \sim k$ does not apply there.

In order to understand why, notice that the excitation probability is a scaling function, $p_k(t) = p(t/\hat{t}, k/\hat{k})$, that is nonzero only up to $k \approx \hat{k} \equiv 1/\sqrt{\tau_Q}$. In this regime of small k , the dispersion (31) is $\omega_k \approx 2\sqrt{\epsilon^2 + k^2} - \epsilon k^2$, where $\epsilon = 1 - g = t/\tau_Q$. With a new integration variable $q = k/\hat{k}$,

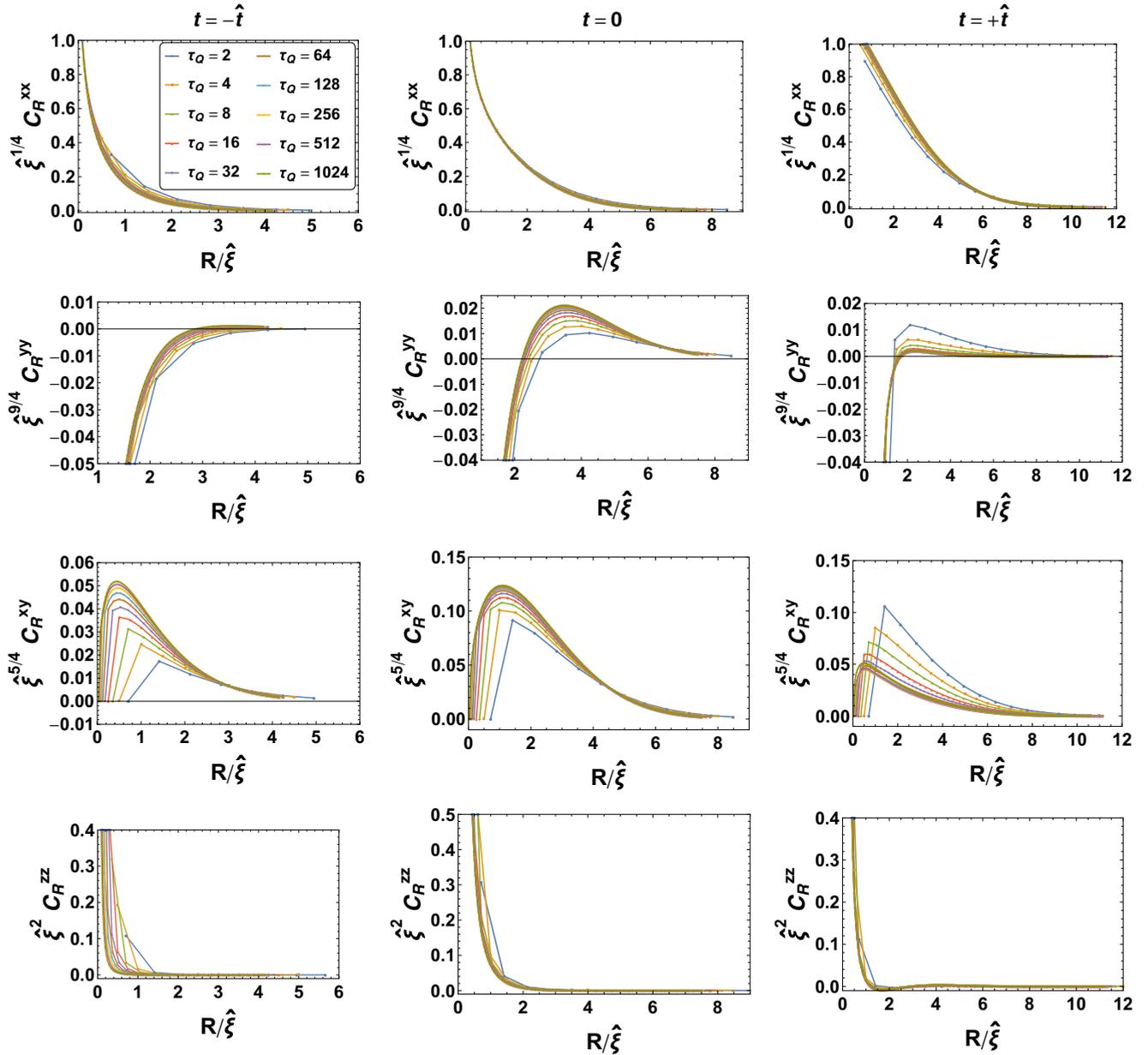


FIG. 7. Spin-spin correlation functions in Eq. (59). The left, middle, and right columns show the correlators before ($t/\hat{t} = -1$), at ($t/\hat{t} = 0$), and after ($t/\hat{t} = +1$) the critical point, respectively. The first row shows the strongest ferromagnetic correlator C_R^{xx} , the second and third ones show C_R^{yy} and C_R^{xy} , respectively, and the bottom one shows the transverse C_R^{zz} . Different colors of the plots correspond to different quench times τ_Q . All plots are rescaled: they are presented as a function of the rescaled distance $R/\hat{\xi} = R/\sqrt{\tau_Q}$ and the correlators are multiplied with $\hat{\xi}^{\Delta_a + \Delta_b} = (\sqrt{\tau_Q})^{\Delta_a + \Delta_b}$. At large distance, $R \gg 1$, the plots in all 12 panels collapse asymptotically with increasing τ_Q , demonstrating the space-time scaling (59) for large enough τ_Q . The collapse does not happen at short distance, $R \approx 1$, where the microscopic lattice constant remains a relevant scale of length and the scaling hypothesis is not expected to hold. The collapsed plots for large τ_Q are the scaling functions $F_C^{ab}(t/\hat{t}, R/\hat{\xi})$ in Eq. (60).

Eq. (55) becomes

$$\frac{W}{N} = \frac{2\hat{k}^2}{\pi} \int_0^\infty dq p\left(\frac{t}{\hat{t}}, q\right) \sqrt{\left(\frac{t}{\hat{t}}\right)^2 + q^2 - \frac{q^2}{\sqrt{\tau_Q}} \left(\frac{t}{\hat{t}}\right)}. \quad (58)$$

In the adiabatic limit, the last term under the square root becomes negligible and the right-hand side becomes $\hat{\xi}^{-2} F_W(t/\hat{t})$,

i.e., a scaling function of t/\hat{t} only. For a given t/\hat{t} , the excitation energy scales like $W/N \sim \hat{\xi}^{-2} = \tau_Q^{-1}$.

This seems to contradict Ref. [55], where the excitation energy at $g = 0$ (proportional to the number of kinks) scales like $W/N \sim \hat{\xi}^{-1} = \tau_Q^{-1/2}$. However, there is no contradiction, since the two scalings compare energies for different τ_Q either at a constant t/\hat{t} or a constant g . At the constant $g = 0$, corresponding to the τ_Q -dependent $t/\hat{t} = \sqrt{\tau_Q}$, we have a flat dispersion $\omega_k = 2$ in Eq. (55) and the excitation energy

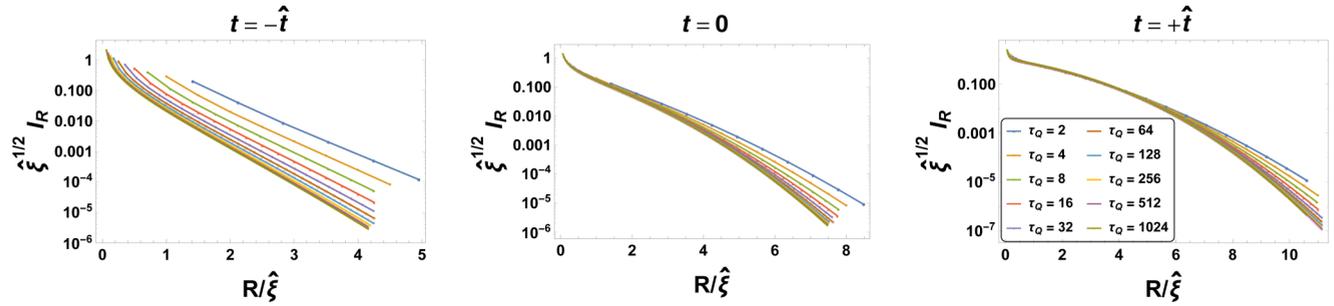


FIG. 8. Mutual information in Eq. (63). The left, middle, and right panels show the mutual information before ($t/\hat{t} = -1$), at ($t/\hat{t} = 0$), and after ($t/\hat{t} = +1$) the critical point, respectively. Different colors of the plots correspond to different quench times τ_Q . All plots are rescaled: they are presented as a function of the rescaled distance $R/\hat{\xi} = R/\sqrt{\tau_Q}$ and the mutual information is multiplied with $\hat{\xi}^{4\Delta_a} = (\sqrt{\tau_Q})^{1/2}$. The plots in all three panels collapse asymptotically with increasing τ_Q , demonstrating the space-time scaling (65) for large enough τ_Q . The collapsed plots for large τ_Q are the scaling function $F_I(t/\hat{t}, R/\hat{\xi})$ in Eq. (65).

is proportional to the number of quasiparticle excitations, $W/N = 4n/N \sim \hat{\xi}^{-1} = \tau_Q^{-1/2}$. For illustration, in Fig. 6 we show the quasiparticle and energy densities as a function of g instead of t/\hat{t} . Everywhere except near the gapless critical points, for a fixed g , the energy scales like $W/N \sim \hat{\xi}^{-1} = \tau_Q^{-1/2}$. This is a remarkable change of perspective, even though the picture away from criticality is sensitive to relevant/nonintegrable perturbations of the Ising model.

E. Scaling in two-spin correlators

The quadratic fermionic correlators are the building blocks for spin correlators:

$$C_R^{ab}(t) \equiv \langle \sigma_n^a \sigma_{n+R}^b \rangle - \langle \sigma_n^a \rangle \langle \sigma_{n+R}^b \rangle. \quad (59)$$

Except for the transverse C^{zz} , they are Pfaffians of matrices whose elements are the fermionic correlators (42) and (43); see Ref. [72].

The KZ scaling implies that

$$C_R^{ab}(t) = \hat{\xi}^{-\Delta_a - \Delta_b} F_C^{ab}(t/\hat{t}, R/\hat{\xi}). \quad (60)$$

Here, Δ_a is the scaling dimension for the operator σ^a . In the Ising chain, we have $\Delta_x = \frac{1}{8}$, $\Delta_y = \frac{9}{8}$, and $\Delta_z = 1$. Figure 7 shows rescaled plots of all nonzero correlators. Their collapse for large enough τ_Q confirms the space-time scaling for large distances $R \gg 1$, where the scaling hypothesis is expected to hold.

F. Scaling in mutual information

The overall strength of spin-spin correlations can be conveniently characterized by mutual information between the two spins. A reduced density matrix for the n th spin is

$$\rho_n^{(1)} = \frac{1}{2}(1_n + \langle \sigma_n^z \rangle \sigma_n^z). \quad (61)$$

A reduced density matrix for spins n and $n+R$ includes their correlations:

$$\rho_{n,n+R}^{(2)} = \rho_n^{(1)} \otimes \rho_{n+R}^{(1)} + \frac{1}{4} \sum_{a,b=1}^3 C_R^{ab} \sigma_n^a \otimes \sigma_{n+R}^b. \quad (62)$$

The correlations contribute to nonzero mutual information between the spins,

$$I_R = S[\rho_n^{(1)}] + S[\rho_{n+R}^{(1)}] - S[\rho_{n,n+R}^{(2)}]. \quad (63)$$

Here, $S[\rho] = -\text{Tr} \rho \ln \rho$ is the von Neumann entropy.

When the correlations C_R^{ab} are weak, for large R or large τ_Q or both, they are a small perturbation to the uncorrelated product $\rho_n^{(1)} \otimes \rho_{n+R}^{(1)}$. To leading order, the mutual information is a quadratic form in C_R 's whose coefficients depend on the transverse magnetization $\langle \sigma^z \rangle$. For slow enough τ_Q , the magnetization can be approximated by its value in the ground state at the critical point, $\langle \sigma^z \rangle \approx 2/\pi$, and it is enough to keep only the dominant term that is quadratic in the strongest correlator C_R^{xx} :

$$\begin{aligned} I_R &\approx \frac{\pi}{8} \left[\frac{2\pi}{\pi^2 - 4} + \text{arctanh}\left(\frac{2}{\pi}\right) \right] (C_R^{xx})^2 \\ &= 0.72 (C_R^{xx})^2. \end{aligned} \quad (64)$$

Consequently, the mutual information should scale as

$$I_R(t) = \hat{\xi}^{-4\Delta_x} F_I(t/\hat{t}, R/\hat{\xi}), \quad (65)$$

where $F_I \propto (F_C^{xx})^2$ is a scaling function. This scaling is demonstrated by the collapsing plots in Fig. 8.

G. Scaling in quantum discord

A convenient measure of quantumness of correlations between spins n and $n+R$ is the quantum discord [71]:

$$\delta_{n|n+R} = \text{Min}_\sigma S[n|\sigma_{n+R}] + S[\rho_{n+R}^{(1)}] - S[\rho_{n,n+R}^{(2)}]. \quad (66)$$

Here,

$$S[n|\sigma_{n+R}] = \sum_{j=\pm 1} p_j S \left[\frac{P_j \rho_{n,n+R}^{(2)} P_j}{p_j} \right], \quad (67)$$

$P_j = (1 + j\sigma_{n+R})/2$ is a projector on the measurement outcome $j = \pm 1$ in the eigenbasis of a Pauli operator σ_{n+R} , and $p_j = \text{Tr} P_j \rho_{n,n+R}^{(2)}$ is a probability of this outcome. In the problem considered in this paper, the discord is symmetric,

$$\delta_{n|n+R} = \delta_{n+R|n} \equiv \delta_R, \quad (68)$$

and the minimum is achieved for $\sigma = \sigma^x$. Not surprisingly, the strongest ferromagnetic correlations are the most classical.

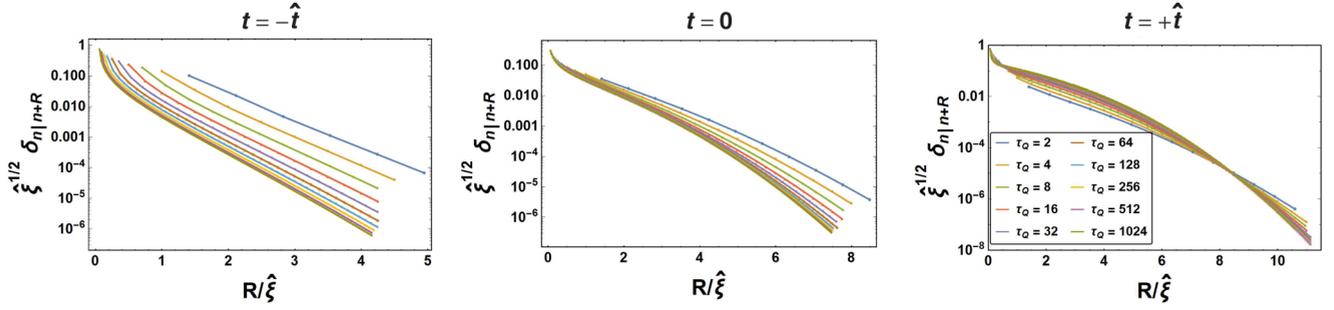


FIG. 9. Quantum discord in Eq. (66). The left, middle, and right panels show the discord before ($t/\hat{t} = -1$), at ($t/\hat{t} = 0$), and after ($t/\hat{t} = +1$) the critical point, respectively. Different colors of the plots correspond to different quench times τ_Q . All plots are rescaled: they are presented as a function of the rescaled distance $R/\hat{\xi} = R/\sqrt{\tau_Q}$ and the discord is multiplied with $\hat{\xi}^{4\Delta_a} = (\sqrt{\tau_Q})^{1/2}$. The plots in all three panels collapse asymptotically with increasing τ_Q , demonstrating the space-time scaling (70) for large enough τ_Q . The collapsed plots for large τ_Q are the scaling function $F_\delta(t/\hat{t}, R/\hat{\xi})$ in Eq. (70).

Like the mutual information, for slow enough τ_Q the discord becomes

$$\begin{aligned} \delta_R &\approx \frac{\pi}{8} \left[\frac{2\pi}{\pi^2 - 4} - \operatorname{arctanh}\left(\frac{2}{\pi}\right) \right] (C_R^{xx})^2 \\ &= 0.12 (C_R^{xx})^2. \end{aligned} \quad (69)$$

This suggests a space-time scaling,

$$\delta_R(t) = \hat{\xi}^{-4\Delta_x} F_\delta(t/\hat{t}, R/\hat{\xi}), \quad (70)$$

that is demonstrated by the collapsing plots in Fig. 9.

H. Scaling in block entropy

In order to go beyond the two-point correlations, one can consider a block of L consecutive spins. Their reduced density matrix is obtained [73] from a correlator matrix,

$$\Pi_L = \begin{pmatrix} A, & B^\dagger \\ B, & 1 - A \end{pmatrix}, \quad (71)$$

where A and B are $L \times L$ Toeplitz matrices $A_{m,n} \equiv \langle c_m c_n^\dagger \rangle = \alpha_{m-n}$ and $B_{m,n} \equiv \langle c_m c_n \rangle = \beta_{m-n}$. The Hermitian Π_L has eigenvalues $0 \leq N_1 \leq \dots \leq N_{2L} \leq 1$, with a symmetry $N_m = 1 - N_{2L+1-m}$. The eigenvalues N_1, \dots, N_L are average occupation numbers for Bogoliubov quasiparticles $\Gamma_1, \dots, \Gamma_L$ localized on the L sites of the block, where we have a Bogoliubov transformation

$$c_n = \sum_{m=1}^L (U_{nm} \Gamma_m + V_{nm}^* \Gamma_m^\dagger). \quad (72)$$

The m th Bogoliubov mode (U_{nm}, V_{nm}) is the eigenvector of Π_L with the eigenvalue N_m .

In this Bogoliubov representation, the reduced density matrix becomes a simple product,

$$\rho_L = \prod_{m=1}^L [N_m |1_m\rangle \langle 1_m| + (1 - N_m) |0_m\rangle \langle 0_m|]. \quad (73)$$

Here, $|1_m\rangle$ ($|0_m\rangle$) is a state with one (zero) quasiparticle annihilated by Γ_m . Consequently, the entropy of entanglement

of the block of L spins with the rest of the lattice is a sum:

$$\begin{aligned} S &= -\operatorname{Tr} \rho_L \ln \rho_L \\ &= -\sum_{m=1}^L [N_m \ln N_m + (1 - N_m) \ln(1 - N_m)] \\ &= -\sum_{m=1}^{2L} N_m \ln N_m. \end{aligned} \quad (74)$$

Interestingly, the last sum is simply $-\operatorname{Tr} \Pi_L \ln \Pi_L$.

Near the critical point in the ground state with a long correlation length ξ , the entropy is $S = \frac{c}{3} \ln \kappa \xi$ for a large block with $L \gg \xi$ and $S = \frac{c}{3} \ln \kappa L$ for a relatively small one with $1 \ll L \ll \xi$. Here, $c = \frac{1}{2}$ is the central charge and $\kappa \simeq 1$ is a nonuniversal constant. With the KZ substitution $\xi \rightarrow \hat{\xi}$, motivated by the adiabatic-impulse approximation, in a dynamical transition we expect [43], respectively, $S = \frac{c}{3} \ln \kappa \hat{\xi}$ and $S = \frac{c}{3} \ln \kappa L$. Beyond this approximation, we allow κ to be a function of the rescaled time t/\hat{t} . This argument suggests a space-time scaling,

$$\frac{S(t, L)}{\frac{c}{3} \ln \kappa(t/\hat{t}) \hat{\xi}} = F_S(t/\hat{t}, L/\hat{\xi}), \quad (75)$$

for large enough τ_Q . Here we assume the normalization $F_S(t/\hat{t}, \infty) = 1$ so that the equation

$$S(t, \infty) = \frac{c}{3} \ln \kappa(t/\hat{t}) \hat{\xi} \equiv S_\infty \quad (76)$$

defines implicitly the function $\kappa(t/\hat{t})$. The scaling is demonstrated by the collapsing plots in Fig. 10. Since the entropy is only logarithmic in τ_Q , the collapse requires much longer quench times than the spin-spin correlators.

I. Scaling in entanglement gap

The entanglement gap is defined as a difference between the two largest coefficients in the Schmidt decomposition between the block of L spins and the rest of the spin chain or, equivalently, between the square roots of the two largest eigenvalues of ρ_L . Since the largest eigenvalues are $(1 - N_1) \dots (1 - N_{L-1})(1 - N_L) > (1 - N_1) \dots (1 - N_{L-1})N_L$, the entangle-

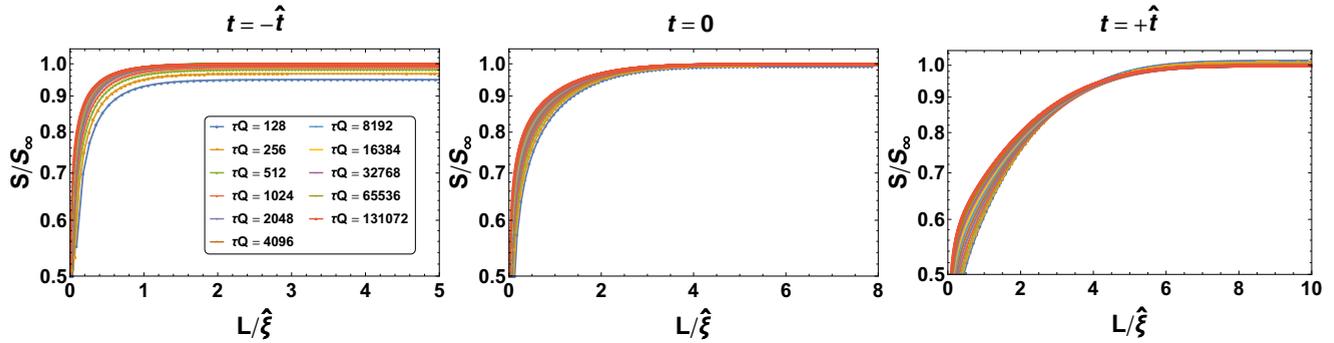


FIG. 10. Entanglement entropy in Eq. (74). The left, middle, and right panels show the entropy before ($t/\hat{t} = -1$), at ($t/\hat{t} = 0$), and after ($t/\hat{t} = +1$) the critical point, respectively. Different colors of the plots correspond to different quench times τ_Q . All plots are rescaled: they are presented as a function of the rescaled size of the block, $L/\hat{\xi} = L/\sqrt{\tau_Q}$, and the entropy is divided by $S_\infty(t/\hat{t}) = \frac{c}{3} \ln k \hat{\xi} = \frac{c}{3} \ln k + \frac{c}{6} \ln \tau_Q$. In consistency with the exact $c = \frac{1}{2}$, the best fits yield $c = 0.524(3), 0.5030(4), 0.478(2)$ at $t/\hat{t} = -1, 0, 1$, respectively. The plots in all three panels collapse asymptotically with increasing τ_Q , demonstrating the space-time scaling (75) for large enough τ_Q . The collapsed plots for large τ_Q are the scaling function $F_S(t/\hat{t}, L/\hat{\xi})$ in Eq. (75).

ment gap reads

$$\Delta\lambda = \sqrt{(1 - N_1) \dots (1 - N_{L-1})} (\sqrt{1 - N_L} - \sqrt{N_L}). \quad (77)$$

According to Ref. [74], in the ground state for a large block with $L \gg \hat{\xi}$, the entanglement gap should scale as

$$\Delta\lambda \sim \xi^{-\beta/\nu}, \quad (78)$$

where $\beta = \frac{1}{8}$ is the critical exponent for the order parameter. In a dynamical transition, we substitute in the above formula ξ with $\hat{\xi}$. Even more generally, for a finite block of size L , we can formulate a scaling law,

$$\Delta\lambda = \hat{\xi}^{-\beta/\nu} F_{\Delta\lambda}(t/\hat{t}, L/\hat{\xi}), \quad (79)$$

expected to hold for slow enough τ_Q . The collapsing plots in Fig. 11 demonstrate this scaling law.

VI. CONCLUSION

We made an extensive overview of the KZ space-time scaling in the quantum Ising chain. We conclude that it is satisfied in the slow quench limit by all of the quantities we have considered. The limit is approached the fastest for the

ferromagnetic correlator. The scaling dimensions proved to be the same as in the static case.

It is tempting to speculate that our conclusion, while for the moment verified only in the Ising chain, may be a useful way of thinking about other quantum phase transitions as well as second-order thermal phase transitions that cannot be probed with exactly solvable models. This would pave the way towards a vast extension of the renormalization from the static equilibrium critical phenomena to the space-time renormalization of phase transition dynamics.

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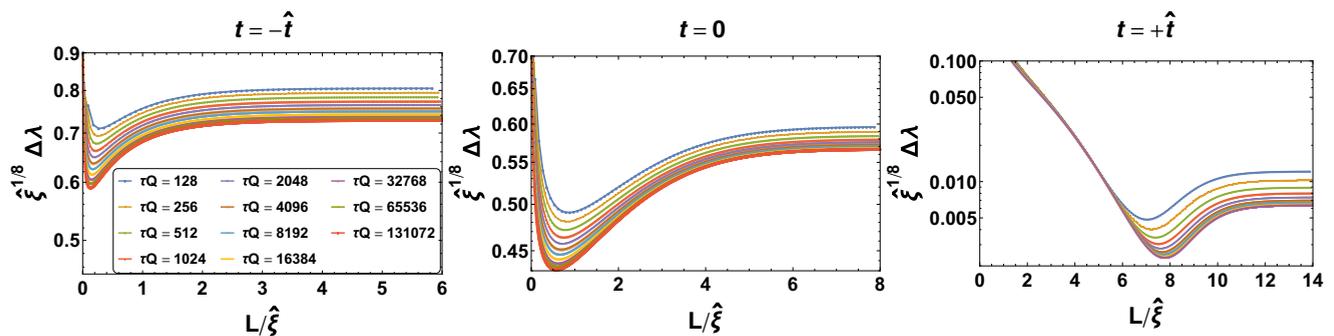


FIG. 11. Entanglement gap in Eq. (77). The left, middle, and right panels show the gap before ($t/\hat{t} = -1$), at ($t/\hat{t} = 0$), and after ($t/\hat{t} = +1$) the critical point, respectively. Different colors of the plots correspond to different quench times τ_Q . All plots are rescaled: they are presented as a function of the rescaled block size $L/\hat{\xi} = L/\sqrt{\tau_Q}$ and the gap is multiplied by $\hat{\xi}^{\beta/\nu} = (\sqrt{\tau_Q})^{1/8}$. The plots in all three panels collapse asymptotically with increasing τ_Q , demonstrating the space-time scaling (79) for large enough τ_Q . The collapsed plots for large τ_Q are the scaling function $F_{\Delta\lambda}(t/\hat{t}, L/\hat{\xi})$ in Eq. (79).

APPENDIX: QUASIMOMENTUM INTEGRALS IN FERMIONIC CORRELATORS

The fermionic correlators (42) and (43) are obtained by numerical integration in MATHEMATICA. In principle, the integrals should be done with the exact solutions (37) in the full integration range $k = 0 \dots \pi$, but it quickly becomes impractical above $\tau_Q \approx 10$. Therefore, we split the range into two. For instance,

$$\beta_R = \int_0^{A\hat{k}} \frac{dk}{\pi} u_k v_k^* \sin kR + \int_{A\hat{k}}^{\pi} \frac{dk}{\pi} u_k v_k^* \sin kR. \quad (\text{A1})$$

The first integral, covering more than the nonadiabatic regime $k = 0 \dots \hat{k}$ for $A > 1$, is done exactly. In the second integral, where the evolution of the Bogoliubov modes is approximately adiabatic, we could approximate the Bogoliubov coefficients by just the positive-frequency adiabatic eigenstate,

$$\begin{pmatrix} u_k \\ v_k \end{pmatrix} \approx \begin{pmatrix} U_k \\ V_k \end{pmatrix}; \quad (\text{A2})$$

compare Eq. (30). However, a much better approximation is obtained at very little expense by also including a first-order perturbative correction,

$$\begin{pmatrix} u_k \\ v_k \end{pmatrix} \approx \frac{1}{\sqrt{1+|B|^2}} \left[\begin{pmatrix} U_k \\ V_k \end{pmatrix} + B \begin{pmatrix} -V_k \\ U_k \end{pmatrix} \right]. \quad (\text{A3})$$

Here, B is an amplitude of excitation to the adiabatic negative-frequency mode,

$$B = \frac{1}{2i\omega_k(g)\tau_Q} \begin{pmatrix} U_k \\ V_k \end{pmatrix}^\dagger \frac{d}{dg} \begin{pmatrix} -V_k \\ U_k \end{pmatrix}. \quad (\text{A4})$$

The results do not depend on A in the range 2..3.

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