

## CONTROLLABILITY AND OBSERVABILITY OF THE DESCRIPTOR LINEAR SYSTEMS REDUCED TO THE STANDARD ONES BY FEEDBACKS

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**Abstract:** In this paper, a new method for the reduction of the descriptor linear systems to standard ones is presented and verified. The method uses a state and/or state derivative feedback of output and output derivative feedback in order to transform the descriptor system into a standard one. The controllability and observability properties of the original descriptor as well as transformed standard systems are proved. Simple numerical examples illustrate the theorems introduced.

**Key words:** controllability and observability, descriptor linear systems, state feedback, state derivative feedback, standard linear systems

### 1. INTRODUCTION

In the theory of linear control systems, the notions of controllability, and observability introduced by Kalman [1, 2], play a fundamental role [3, 4, 5]. Some recent developments on this crucial notions have been presented in the papers [5, 6, 7], and the references therein. It is well-known that the dynamical properties of linear controllable systems can be modified by the use of state feedbacks [8,9,10]. On the other hand the descriptor (aka singular or implicit) systems have been subject to intensive investigation in recent years (see e.g. [11, 12, 13], for details). In this paper, we shall concentrate on controllability, and its dual concept observability of descriptor linear systems. The main idea of the paper is to prove that controllability, and observability of linear descriptor systems with constant coefficients, and with appropriately defined linear feedbacks depending both on state vector and its derivatives, is equivalent to controllability, and observability of standard systems. The paper is organized as follows: in Section (2) the controllability and in Section (3) the observability of the descriptor linear systems with state and state derivative feedbacks are analyzed. The controllability and observability of the descriptor linear systems with state derivative feedbacks are investigated in Section (4). Concluding remarks are given in Section (5).

### 2. CONTROLLABILITY OF DESCRIPTOR LINEAR SYSTEMS WITH FEEDBACKS

Let us consider the descriptor, finite-dimensional, linear continuous-time system:

$$E\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx \tag{1b}$$

where  $t \in [0, t_f]$ , and  $x = x(t) \in \mathbb{R}^n, u = u(t) \in \mathbb{R}^m, y = y(t) \in \mathbb{R}^p$  are the state, input and output vectors, respectively, and  $E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}$  are constant matrices. It is assumed that:

$$\det[Es - A] \neq 0 \tag{2}$$

In this case the equation (1a) has a unique solution, given in [4].

Definition 1 ([3]). The system (1a) is called controllable if for any initial state  $x(0) \in \mathbb{R}^n$  and any finite state  $x_f \in \mathbb{R}^n$  there exists an input  $u(t), t \in [0, t_f]$  such that  $x(t_f) = x_f$ .

Theorem 1. The system (1a) is controllable if and only if:

$$\text{rank}[Es - A, B] = n \text{ for all } s \in \mathbb{F} \tag{3a}$$

$$\text{rank}[E, B] = n \tag{3b}$$

where  $\mathbb{F}$  is the field of complex numbers.

Proof of this theorem is given in [3].

Let us consider the system (1) with the linear feedback depending both on state vector and its derivatives (see Fig. 1)

$$u = v - K_1x - K_2\dot{x} \tag{4}$$

where  $v = v(t) \in \mathbb{R}^m$  is the new input and  $K_1, K_2 \in \mathbb{R}^{m \times n}$ . Substituting linear feedback (4) into (1) we obtain:

$$(E + BK_2)\dot{x} = (A - BK_1)x + Bv \tag{5}$$

Note that if the condition (3) is satisfied then from the equation:

$$E + BK_2 = [E, B] \begin{bmatrix} I \\ K_2 \end{bmatrix} \tag{6}$$

we may find the matrix  $K_2$  such that the matrix  $E + BK_2$  is nonsingular and from equality (5) we obtain

$$\begin{aligned} \dot{x} &= [E + BK_2]^{-1}[(A - BK_1)x + Bv] \\ &= (\bar{A} - \bar{B}K_1)x + \bar{B}v \end{aligned} \tag{7}$$

where

$$[\mathbb{E} + \mathbb{B}\mathbb{K}_2]^{-1}(\mathbb{A} - \mathbb{B}\mathbb{K}_1) = \bar{\mathbb{A}} - \bar{\mathbb{B}}\mathbb{K}_1, \quad (8)$$

$$\bar{\mathbb{A}} = [\mathbb{E} + \mathbb{B}\mathbb{K}_2]^{-1}\mathbb{A}, \quad \bar{\mathbb{B}} = [\mathbb{E} + \mathbb{B}\mathbb{K}_2]^{-1}\mathbb{B}$$

Therefore the following theorem has been proved.

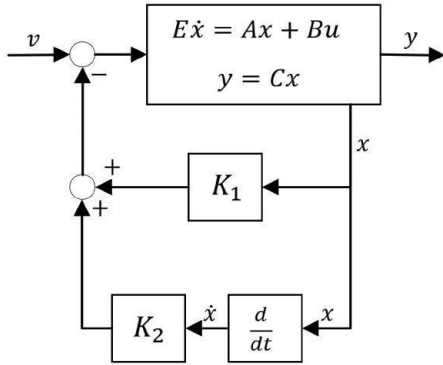


Fig. 1. Descriptor system with state and state derivative feedbacks

**Theorem 2.** If the system (1a) is controllable (the condition (3) is satisfied) then there exists a matrix  $\mathbb{K}_2$  such that the matrix on the left-hand side of equation (5) (i.e.  $\mathbb{E} + \mathbb{B}\mathbb{K}_2$ ) is nonsingular and the descriptor system (1a) can be reduced to the standard one (7).

Moreover, taking into account the above relations, the following result on controllability can be proven.

**Theorem 3.** The standard system (7) is controllable if and only if the descriptor system (1a) with feedback (4) is controllable.

Proof: From the equality:

$$[\mathbb{A} + \mathbb{B}\mathbb{B}_2, B] = [\mathbb{A}, \mathbb{B}] \begin{bmatrix} \mathbb{I}_n & 0 \\ \mathbb{K}_2 & \mathbb{I}_m \end{bmatrix} \quad (9)$$

it follows that

$$\text{rank}[\mathbb{A} + \mathbb{B}_2, B] = \text{rank}[\mathbb{A}, \mathbb{B}] \quad (10)$$

for any matrix  $\mathbb{K}_2$ . Note that

$$\text{rank}[\mathbb{I}_n s - \bar{\mathbb{A}} + \bar{\mathbb{B}}\mathbb{K}_1, \bar{\mathbb{B}}] = \text{rank}[\mathbb{I}_n s - \bar{\mathbb{A}}, \bar{\mathbb{B}}] \begin{bmatrix} \mathbb{I}_n & 0 \\ \mathbb{K}_1 & \mathbb{I}_m \end{bmatrix} \quad (11)$$

Therefore, the pair  $(\bar{\mathbb{A}} - \bar{\mathbb{B}}\mathbb{K}_1, \bar{\mathbb{B}})$  is controllable if and only if the pair  $\bar{\mathbb{A}}, \bar{\mathbb{B}}$  is controllable. This completes the proof.

**Example 1.** For the system (1) with

$$\mathbb{E} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{A} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbb{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbb{C} = [1 \ 0 \ 0] \quad (12)$$

compute the matrix  $\mathbb{K}_2$  such that  $\det[\mathbb{E} + \mathbb{B}\mathbb{K}_2] \neq 0$ , and the matrices  $\bar{\mathbb{A}}$ , and  $\bar{\mathbb{B}}$ . Using (8) and (1) we obtain for  $\mathbb{K}_2 = [1 \ 0 \ 0]$

$$\bar{\mathbb{A}} = [\mathbb{E} + \mathbb{B}\mathbb{K}_2]^{-1}\mathbb{A} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [1 \ 0 \ 0] \right\}^{-1} \times$$

$$\times \begin{bmatrix} -2 & 0 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 1 \\ 0 & -3 & 0 \end{bmatrix} \quad (13a)$$

$$\bar{\mathbb{B}} = [\mathbb{E} + \mathbb{B}\mathbb{K}_2]^{-1} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [1 \ 0 \ 0] \right\}^{-1} \times$$

$$\times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (13b)$$

The system (7) with (13) is controllable since by Theorem 1

$$\text{rank}[\mathbb{E}s - \bar{\mathbb{A}}, \bar{\mathbb{B}}] = \text{rank} \begin{bmatrix} 2 & s & -1 & 0 \\ 0 & 3 & s & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = 3 \quad (14)$$

for  $s \in \mathbb{F}$ , where  $\mathbb{F}$  is the field of complex numbers. and

$$\text{rank}[\mathbb{E}, \bar{\mathbb{B}}] = \text{rank} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 3 \quad (15)$$

The system with (13) is also controllable since

$$\text{rank}[\bar{\mathbb{B}}, \bar{\mathbb{A}}\bar{\mathbb{B}}, \bar{\mathbb{A}}^2\bar{\mathbb{B}}] = \text{rank} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 3 \quad (16)$$

This confirms Theorems 2 and 3.

### 3. OBSERVABILITY OF DESCRIPTOR LINEAR SYSTEMS WITH FEEDBACKS

**Definition 2** ([3]). The system (1) is called observable if knowing the input  $u(t)$ , and the output  $y(t)$  for  $t \in [0, t_f]$ , it is possible to find the initial state of the system.

**Theorem 4.** The system (1a) and (1b) is observable if and only if

$$\text{rank} \begin{bmatrix} \mathbb{E}s - \mathbb{A} \\ \mathbb{C} \end{bmatrix} = n \text{ for all } s \in \mathbb{F} \quad (17a)$$

$$\text{rank} \begin{bmatrix} \mathbb{E} \\ \mathbb{C} \end{bmatrix} = n \quad (17b)$$

Proof of this theorem is given in [3].

Let us consider the system (1) with the output feedback (see Fig. 2):

$$u = v - \mathbb{F}_1 y - \mathbb{F}_2 \dot{y} \quad (18)$$

where  $v = v(t) \in \mathbb{R}^m$  is the new input and  $\mathbb{F}_1, \mathbb{F}_2 \in \mathbb{R}^{m \times p}$ . Substituting (1b) into (18) we obtain

$$u = v - \mathbb{F}_1 \mathbb{C}x - \mathbb{F}_2 \mathbb{C}\dot{x} \quad (19)$$

Substitution of (19) into (1a) yields

$$(\mathbb{E} + \mathbb{B}\mathbb{F}_2\mathbb{C})\dot{x} = (\mathbb{A} - \mathbb{B}\mathbb{F}_1\mathbb{C})x + \mathbb{B}v \quad (20)$$

Note that if the condition (17a) is satisfied then from the equality

$$\mathbb{E} + \mathbb{B}\mathbb{F}_2\mathbb{C} = [\mathbb{I}_n, \mathbb{B}\mathbb{F}_2] \begin{bmatrix} \mathbb{E} \\ \mathbb{C} \end{bmatrix} \quad (21)$$

we may find (compute) the matrix  $\mathbb{F}_2$  such that the matrix  $\mathbb{E} + \mathbb{B}\mathbb{F}_2\mathbb{C}$  is nonsingular and from (20) we obtain

$$\dot{x} = [\mathbb{E} + \mathbb{B}\mathbb{F}_2\mathbb{C}]^{-1}[(\mathbb{A} - \mathbb{B}\mathbb{F}_1\mathbb{C})x + \mathbb{B}v]$$

$$= (\hat{\mathbb{A}} - \hat{\mathbb{B}}_1\mathbb{C})x + \hat{\mathbb{B}}v \quad (22)$$

where

$$\begin{aligned} \hat{A} &= [E + BF_2C]^{-1}A \\ \hat{B} &= [E + BF_2C]^{-1}B \end{aligned} \quad (23)$$

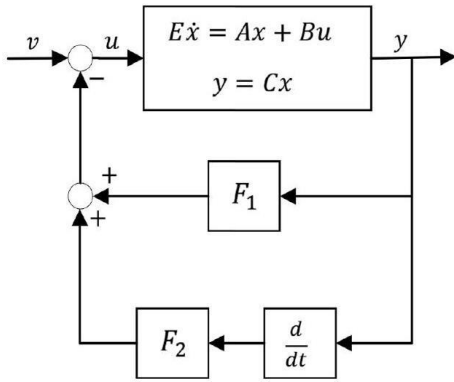


Fig. 2. Descriptor system with output and output derivative feedbacks

Therefore, from (22) and (23) the following Theorem has been proved:

Theorem 5. If the system (1) is observable (the condition (17a) is satisfied), and

$$\text{rank}[E + BF_2C] = \min \left\{ \text{rank} \begin{bmatrix} E \\ C \end{bmatrix}, \text{rank}[E, B] \right\} \quad (24)$$

then there exists a matrix  $F_2$  such that the matrix (21) is nonsingular and the descriptor system (1) can be reduced to the standard one (22).

Theorem 6. The standard system (22) is observable if and only if the descriptor system (1) with feedback (19) is observable.

Proof: From the equality:

$$[E + BF_2C] = [I_n, BF_2] \begin{bmatrix} E \\ C \end{bmatrix} \quad (25)$$

it follows that

$$\text{rank}[E + BF_2C] = \text{rank} \begin{bmatrix} E \\ C \end{bmatrix} \quad (26)$$

for any matrix  $F_2$ . Note that

$$\text{rank} \begin{bmatrix} I_n s - \hat{A} + \hat{B}F_1C \\ \hat{C} \end{bmatrix} = \text{rank} \begin{bmatrix} I_n & \hat{B}_1 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} I_n s - \hat{A} \\ C \end{bmatrix} \quad (27)$$

Therefore, the pair  $(\hat{A} - \hat{B}F_1C, \hat{C})$  is observable if and only if the pair  $\hat{A}, C$  is observable.

This completes the proof.

Example 2. Consider the system (1) with (18). Compute the matrix  $F_2$  such that  $\det[E + BF_2C] \neq 0$  and the matrices  $\hat{A}$ , and  $\hat{B}$ . The system (22) with the matrices  $E, A$ , and  $C$  given by (1) is observable since

$$\text{rank} \begin{bmatrix} Es - A \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} 2 & s & -1 \\ 0 & 3 & s \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} = 3 \text{ for } s \in \mathbb{F} \quad (28)$$

and

$$\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = 3 \quad (29)$$

The system (22) with matrices

$$F_2 = 1 \quad (30)$$

and

$$\hat{A} = [E + BF_2C]^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \hat{C} = C \quad (31)$$

is also observable, since

$$\text{rank} \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \\ \hat{C}\hat{A}^2 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = 3 \quad (32)$$

#### 4. CONTROLLABILITY AND OBSERVABILITY OF THE DESCRIPTOR MODELS UNDER THE STATE DERIVATIVE FEEDBACK

In this section we will show that under some assumptions and using only a state derivative feedback, a controllable and observable descriptor system can be transformed to a standard controllable and observable system. This result can be formulated in the following theorem:

Theorem 7. If the system (1) satisfies the assumptions

$$E = \begin{bmatrix} 0 & E_1 \\ 0 & 0 \end{bmatrix}, \det E_1 \neq 0, E_1 \in \mathbb{R}^{(n-m) \times (n-m)} \quad (33)$$

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \det A_1 \neq 0, \det A_3 \neq 0,$$

$$A_1 \in \mathbb{R}^{m \times m}, A_3 \in \mathbb{R}^{(n-m) \times (n-m)} \quad (34, 35)$$

$$B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \det B_2 \neq 0, C = [C_1 \quad 0] \det C_1 \neq 0,$$

$$B_2, C_1 \in \mathbb{R}^{m \times m}$$

then

$$\text{rank}[E \quad B] = n, \text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n \quad (36)$$

and there exists a state derivative feedback  $u = v - \bar{K}\dot{x}$ , where  $\bar{K} \in \mathbb{R}^{m \times n}$ ,  $\bar{K} = [\bar{K} \quad 0]$  such that  $\bar{K} \in \mathbb{R}^{m \times m}$  and  $\det(B_2\bar{K}) \neq 0$ , the matrix

$$\bar{A} = [E + B\bar{K}]^{-1}A = \begin{bmatrix} 0 & (B_2\bar{K})^{-1}A_3 \\ E_1^{-1}A_1 & E_1^{-1}A_2 \end{bmatrix} \quad (37)$$

is nonsingular,

$$\bar{B} = [E + B\bar{K}]^{-1}B = \begin{bmatrix} (B_2\bar{K})^{-1}B_2 \\ 0 \end{bmatrix} \quad (38)$$

and the pair  $(\bar{A}, \bar{B})$  is controllable, while the pair  $(\bar{A}, \bar{C})$  is observable.

Proof: From (33), and (35), we have

$$\text{rank}[E \quad B] = \text{rank} \begin{bmatrix} 0 & E_1 & 0 \\ 0 & 0 & B_2 \end{bmatrix} = \text{rank } E_1 + \text{rank } B_2 = n \quad (39)$$

and

$$\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & B_1 \\ 0 & 0 \\ C_1 & 0 \end{bmatrix} = \text{rank } E_1 + \text{rank } C_1 = n \quad (40)$$

Using (33), (34), and (35), we obtain

$$\text{rank}[Es - A \quad B] = \text{rank} \begin{bmatrix} -A_1 & E_1 s - A_2 & 0 \\ 0 & -A_3 & B_2 \end{bmatrix} = n \quad (41)$$

since  $\det A_1 \neq 0$ , and  $\det A_3 \neq 0$ .

Similarly,

$$\text{rank} \begin{bmatrix} Es - A \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} -A_1 & E_1 s - A_2 \\ 0 & -A_3 \\ C_1 & 0 \end{bmatrix} = n \quad (42)$$

since  $\det A_1 \neq 0$ , and  $\det A_3 \neq 0$ .

Moreover,

$$\begin{aligned} \bar{A} &= [E + B\bar{K}]^{-1}A = \begin{bmatrix} 0 & E_1 \\ B_2 K & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} = \\ &= \begin{bmatrix} 0 & (B_2 K)^{-1} \\ E_1^{-1} & 0 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} = \\ &= \begin{bmatrix} 0 & (B_2 K)^{-1} A_3 \\ E_1^{-1} A_1 & 2E_1^{-1} A_2 \end{bmatrix} \end{aligned} \quad (43)$$

$$\begin{aligned} \bar{B} &= [E + B\bar{K}]^{-1}B = \begin{bmatrix} 0 & E_1 \\ B_2 K & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \\ &= \begin{bmatrix} 0 & (B_2 K)^{-1} \\ E_1^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} (B_2 K)^{-1} B_2 \\ 0 \end{bmatrix} \end{aligned} \quad (44)$$

$$\begin{aligned} \bar{C} &= C[E + B\bar{K}]^{-1} = [C_1 \quad 0] \begin{bmatrix} 0 & E_1 \\ B_2 K & 0 \end{bmatrix}^{-1} = \\ &= [C_1 \quad 0] \begin{bmatrix} 0 & (B_2 K)^{-1} \\ E_1^{-1} & 0 \end{bmatrix} = \\ &= [0 \quad C_1 (B_2 K)^{-1}]. \end{aligned} \quad (45)$$

Thus, the pair  $(\bar{A}, \bar{B})$  is controllable because

$$\begin{aligned} &\text{rank}[Is - \bar{A} \quad \bar{B}] = \\ &= \text{rank} \begin{bmatrix} Is & -(B_2 K)^{-1} A_3 & (B_2 K)^{-1} B_2 \\ -E_1^{-1} A_1 & Is - E_1^{-1} A_2 & 0 \end{bmatrix} = n \end{aligned} \quad (46)$$

since

$$\text{rank} \begin{bmatrix} 0 & (B_2 K)^{-1} B_2 \\ -E_1^{-1} A_1 & 0 \end{bmatrix} = n$$

The proof of the observability of the pair  $(\bar{A}, \bar{C})$  is similar i.e.

$$\text{rank} \begin{bmatrix} Is - \bar{A} \\ \bar{C} \end{bmatrix} = \text{rank} \begin{bmatrix} Is & -(B_2 K)^{-1} A_3 \\ -E_1^{-1} A_1 & Is - E_1^{-1} A_2 \\ 0 & C_1 (B_2 K)^{-1} \end{bmatrix} = n \quad (47)$$

$$\text{rank} \begin{bmatrix} 0 & (B_2 K)^{-1} A_3 \\ -E_1^{-1} A_1 & -E_1^{-1} A_2 \end{bmatrix} = n$$

This completes the proof.

## 5. CONCLUSIONS

In this paper, the controllability and observability of descriptor linear, continuous-time, finite dimensional systems with constant coefficients have been discussed. Using pure algebraic methods taken directly from the theory of matrices, it was proved that the controllability and observability of descriptor systems with suitably defined linear state and/or state derivative or output and output derivative feedbacks are equivalent to the controllability and observability of standard systems. It should be pointed out that the

proofs provided in the present research make use of results that are already ascertained from the literature.

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