Sub- and super-fidelity as bounds for quantum fidelity

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Quantum states and fidelity

Quantum state is an operator $\rho : \mathcal{H}_N \rightarrow \mathcal{H}_N$, which is positive semi-definite ($\rho \geq 0$) and normalised ($\text{tr}\rho = 1$). We denote by $\Omega_N \subset \mathcal{M}_N$ the space of density matrices on $\mathbb{C}^N$.

We define the fidelity between two states as

$$F(\rho_1, \rho_2) = \left(\text{tr}|\sqrt{\rho_1}\sqrt{\rho_2}|\right)^2 = ||\rho_1^{1/2}\rho_2^{1/2}||^2_1,$$

where $|| \cdot ||_1$ is a trace norm, i.e. $||A||_1 = \text{tr}|A| = \sum_{i=1}^{N} \sigma_i(A)$.

In the case of two pure states $\rho_1 = |\phi\rangle\langle\phi|, \rho_2 = |\psi\rangle\langle\psi|$ we have $F(\rho_1, \rho_2) = |\langle\psi|\phi\rangle|^2$. 

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Sub- and super-fidelity as bounds for quantum fidelity
Fidelity has few nice properties

- **Bounds:** $0 \leq F(\rho_1, \rho_2) \leq 1$. Furthermore $F(\rho_1, \rho_2) = 1$ iff $\rho_1 = \rho_2$, while $F(\rho_1, \rho_2) = 0$ iff $\text{supp}(\rho_1) \perp \text{supp}(\rho_2)$.

- **Symmetry:** $F(\rho_1, \rho_2) = F(\rho_2, \rho_1)$.

- **Unitary invariance:** $F(\rho_1, \rho_2) = F(U\rho_1U^\dagger, U\rho_2U^\dagger)$, for any unitary operator $U$.

- **Concavity:**
  \[
  F(\rho, a\rho_1 + (1-a)\rho_2) \geq aF(\rho, \rho_1) + (1-a)F(\rho, \rho_2), \text{ for } a \in [0,1].
  \]

- **Multiplicativity:** $F(\rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4) = F(\rho_1, \rho_3)F(\rho_2, \rho_4)$.

- **Joint concavity:** $\sqrt{F(a\rho_1 + (1-a)\rho_2, a\rho_1' + (1-a)\rho_2')} \geq a\sqrt{F(\rho_1, \rho_1')} + (1-a)\sqrt{F(\rho_2, \rho_2')}$, for $a \in [0,1]$. 
Classical counterpart

Fidelity between two diagonal operators is equal to the Bhattacharyya coefficient $B$ for their eigenvalues.

$$\sqrt{F(\text{diag}(\rho_1), \text{diag}(\rho_2))} = B(p, q) = \sum_{i=1}^{n} \sqrt{p_i q_i}$$

Here $p$ and $q$ are eigenvalues of $\rho_1$ and $\rho_2$ respectively.
We start our analysis of fidelity by expressing it in terms of eigenvalues \( \lambda_i, \ i = 1, \ldots, N \) of the (positive) matrix \( \sqrt{\rho_1^{1/2} \rho_2 \rho_1^{1/2}} \). Using the fact that matrix \( \rho_1 \rho_2 \) is similar to matrix \( \sqrt{\rho_1 \rho_2} \sqrt{\rho_2} \) one can write

\[
\sqrt{F(\rho_1, \rho_2)} = \text{tr} \sqrt{\sqrt{\rho_1 \rho_2} \sqrt{\rho_1}} = \sum_{i=1}^{N} \lambda_i,
\]

and since \( \text{tr} \rho_1 \rho_2 = \text{tr} \sqrt{\rho_1 \rho_2} \sqrt{\rho_1} = \sum_{i=1}^{N} \lambda_i^2 \) by squaring the above we get

\[
F(\rho_1, \rho_2) = \left( \sum_{i=1}^{N} \lambda_i \right)^2 = \text{tr} \rho_1 \rho_2 + 2 \sum_{i<j} \lambda_i \lambda_j.
\]
Elementary symmetric functions

For a given matrix $X \in \mathcal{M}_N$ with eigenvalues $\lambda_1, \ldots, \lambda_N$ we define elementary symmetric function $s_m(X)$ as $s_m(\lambda_1, \ldots, \lambda_N)$.

For example

$$s_2(X) = \sum_{i<j} \lambda_i \lambda_j,$$

$$s_3(X) = \sum_{i<j<k} \lambda_i \lambda_j \lambda_k.$$

Using this notion we can write the fidelity as

$$F(\rho_1, \rho_2) = \text{tr}\rho_1\rho_2 + 2s_2(\sqrt[4]{\rho_1\rho_2}\sqrt[4]{\rho_1}).$$
Lower bound by Uhlmann

In his unpublished work Uhlmann suggested an inequality

\[ F(\rho_1, \rho_2) \geq \text{tr}\rho_1\rho_2 + \sqrt{2}\sqrt{\text{tr}\rho_1\rho_2^2} - \text{tr}\rho_1\rho_2\rho_1\rho_2. \]

We define sub-fidelity as

\[ E(\rho_1, \rho_2) = \rho_1\rho_2 + \sqrt{2}\sqrt{s_2(\rho_1\rho_2)}. \]

Using elementary symmetric functions this quantity can be represented as

\[ E(\rho_1, \rho_2) = \text{tr}\rho_1\rho_2 + 2\sqrt{s_2(\rho_1\rho_2)}. \]
Super-fidelity

We can introduce upper bound which is complementary to sub-fidelity\(^1\)

\[
F(\rho_1, \rho_2) \leq \text{tr}\rho_1\rho_2 + \sqrt{(1 - \text{tr}\rho_1^2)(1 - \text{tr}\rho_2^2)}.
\]

Again we can use elementary symmetric function to get compact expression for super-fidelity

\[
G(\rho_1, \rho_2) = \text{tr}\rho_1\rho_2 + \sqrt{(1 - \text{tr}\rho_1^2)(1 - \text{tr}\rho_2^2)} = \text{tr}\rho_1\rho_2 + 2\sqrt{s_2(\rho_1)s_2(\rho_2)}.
\]

thus we have

\[
s_2(\sqrt{\sqrt{\rho_1\rho_2}\sqrt{\rho_1}}) \leq \sqrt{s_2(\rho_1)s_2(\rho_2)}
\]

Two inequalities $E(\rho_1, \rho_2) \leq F(\rho_1, \rho_2) \leq G(\rho_1, \rho_2)$ can be written in a compact way using elementary symmetric functions

$$\sqrt{s_2(\rho_1 \rho_2)} \leq s_2(\sqrt[\frac{1}{2}]{\rho_1 \rho_2} \sqrt[\frac{1}{2}]{\rho_1}) \leq \sqrt{s_2(\rho_1)s_2(\rho_2)}$$

- If one of the states is pure we have equality $E = F = G$.
- Moreover these quantities coincide for one-qubit states ($N = 2$).
Properties of sub- and super-fidelity

Sub- and super-fidelity share some properties with fidelity

i’) **Bounds:** $0 \leq E(\rho_1, \rho_2) \leq 1$ oraz $0 \leq G(\rho_1, \rho_2) \leq 1$.

ii’) **Symmetry:** $E(\rho_1, \rho_2) = E(\rho_2, \rho_1)$ and $G(\rho_1, \rho_2) = G(\rho_2, \rho_1)$.

iii’) **Unitary invariance:** $E(\rho_1, \rho_2) = E(U\rho_1 U^\dagger, U\rho_2 U^\dagger)$ and $G(\rho_1, \rho_2) = G(U\rho_1 U^\dagger, U\rho_2 U^\dagger)$, for any unitary $U$.

iv’) **Concavity:** Sub- and super-fidelity are concave,

$$E(A, \alpha B + (1 - \alpha) C) \geq \alpha E(A, B) + (1 - \alpha) E(A, C),$$

$$G(A, \alpha B + (1 - \alpha) C) \geq \alpha G(A, B) + (1 - \alpha) G(A, C).$$

v’) **Super-fidelity** (just like $\sqrt{F}$) is **jointly concave** in its two arguments

$$\sqrt{F}(a\rho_1 + (1-a)\rho_2, a\rho_1' + (1-a)\rho_2') \geq a\sqrt{F}(\rho_1, \rho_1') + (1-a)\sqrt{F}(\rho_2, \rho_2')$$

for $a \in [0, 1]$. 


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Sub- and super-fidelity as bounds for quantum fidelity
Both $E$ and $G$ are not multiplicative. On the other hand
Properties of sub- and super-fidelity
Properties for tensor products

Both $E$ and $G$ are not multiplicative. On the other hand

vi') Super-fidelity is super-multiplicative:

$$G(\rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4) \geq G(\rho_1, \rho_3) G(\rho_2, \rho_4),$$
Properties of sub- and super-fidelity

Properties for tensor products

Both $E$ and $G$ are not multiplicative. On the other hand

vi′) Super-fidelity is super-multiplicative:

$$G(\rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4) \geq G(\rho_1, \rho_3)G(\rho_2, \rho_4),$$

vii′) Sub-fidelity is sub-multiplicative

$$E(\rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4) \leq E(\rho_1, \rho_3)E(\rho_2, \rho_4).$$
If the density matrices $\rho_p$ and $\rho_q$ commute, discussed bound can be expressed in terms of respective eigenvalues $\{p_i\}_{i=1}^N$ and $\{q_i\}_{i=1}^N$:

$$E(\rho_p, \rho_q) = \sum_{i=1}^N p_i q_i + \sqrt{2 \left[ \left( \sum_{i=1}^N p_i q_i \right)^2 - \sum_{i=1}^N p_i^2 q_i^2 \right]},$$

$$F(\rho_p, \rho_q) = \left( \sum_{i=1}^N \sqrt{p_i q_i} \right)^2,$$

$$G(\rho_p, \rho_q) = \sum_{i=1}^N p_i q_i + \sqrt{\left( 1 - \sum_{i=1}^N p_i^2 \right) \left( 1 - \sum_{i=1}^N q_i^2 \right)}.$$
Mixed states

To get some feeling about behaviour of $E$ and $G$ we calculate them for states of the form

$$\rho_a = a |\psi\rangle\langle\psi| + (1 - a) I / N.$$

where $|\psi\rangle$ is an arbitrary pure state.

For $\rho_* := I / N$ we get

$$F(\rho_a, \rho_*) = \frac{1}{N^2} \left( \sqrt{(N - 1)a + 1} + (N - 1)\sqrt{1 - a} \right)^2,$$

and sub- and super-fidelity are expressed as

$$E(\rho_a, \rho_*) = \frac{1}{N} + \sqrt{2} \frac{1}{N} \sqrt{1 - \frac{1}{N}} \sqrt{1 - a^2},$$

$$G(\rho_a, \rho_*) = \frac{1}{N} + \left( 1 - \frac{1}{N} \right) \sqrt{1 - a^2}.$$
Comparison of sub–fidelity $E$, fidelity $F$ and super–fidelity $G$. 

\[ F(a)^i \]

\begin{align*}
\text{a) } N &= 2 \\
E &= F = G \\
\text{b) } N &= 3 \\
\text{c) } N &= 4 \\
\text{d) } N &= 5 \\
E &\quad F &\quad G
\end{align*}

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Difference $G - F$ and $E - F$

$F$ and $G$ coincide if one of the states is pure, but it is natural to ask how big the difference $G - F$ might be. Let us use the Hilbert space of dimension $N = 2M$ and states

$$\rho_1 = \frac{2}{N} \text{diag}(1, \ldots, 1, 0, \ldots, 0)$$

and

$$\rho_2 = \frac{2}{N} \text{diag}(0, \ldots, 0, 1, \ldots, 1).$$
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Let us use the Hilbert space of dimension $N = 2M$ and states

$$\rho_1 = \frac{2}{N}\text{diag}(1,\ldots,1,0,\ldots,0) \quad \text{and} \quad \rho_2 = \frac{2}{N}\text{diag}(0,\ldots,0,1,\ldots,1).$$

Since they are supported by orthogonal subspaces their fidelity vanishes, $F(\rho_1, \rho_2) = 0$. On the other hand their super–fidelity is equal to

$$G(\rho_1, \rho_2) = \frac{N - 2}{N},$$

and the difference $F - G$ can be arbitrarily close to 1.
Maximal difference

Max difference between fidelity and sub- and super-fidelity for random states in dimensions $N = 2, 3, \ldots, 62$. 

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Average difference

On average situation looks somehow better.

Average difference between fidelity and sub- and super-fidelity for some values of $N \in [2, 62]$. 

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Sub- and super-fidelity as bounds for quantum fidelity
Super-fidelity and trace distance

For any $\rho_1, \rho_2 \in \Omega_N$ super-fidelity and trace distance are related by the inequality\(^2\)

$$1 - G(\rho_1, \rho_2) \leq D_{tr}(\rho_1, \rho_2)$$

Probability of error for distinguishing two density matrices $\rho_1, \rho_2 \in \Omega_N$ is expressed by the trace distance as

$$P_E(\rho_1, \rho_2) = \frac{1}{2}(1 - D_{tr}(\rho_1, \rho_2)).$$

Using the above inequalities we can write

$$\frac{1}{2} G(\rho_1, \rho_2) \geq P_E(\rho_1, \rho_2)$$

Experimental setup for measuring super-fidelity

We use fact that \( \text{tr} V_{12} \rho_1 \otimes \rho_2 = \text{tr} \rho_1 \rho_2 \) where \( V_{12} \) is a SWAP operator. \( V_{12} = P_{12}^+ - P_{12}^- \) is hermitian and thus represents an observable.
To measure \( G \) we need a source which creates pairs \( \rho_i \otimes \rho_j \), \( i, j = 1, 2 \).
The probability of measuring \( P_{12}^- \) reads \( p_{ij}^- = \text{tr} P_{12}^- \rho_i \otimes \rho_j \) and using it we can write

\[
G = 1 - 2(\rho_{12}^- - \sqrt{\rho_{11}^- \rho_{22}^-})
\]
Probability of the event that both detectors click is equal to $p_{ij}^-$. On detectors clicks with $p_{ij}^+ = 1 - p_{ij}^-$. Beam-splitter projects on $P^-$ or $P^+$.

The experimental setup is in this case very simple.³

³F. A. Bovino et all, PRL 95, 240407 (2005)
**Computational efficiency**

*E and G are much easier to calculate than fidelity* \( F \). To compute any of these bounds it is enough to evaluate three traces only.

\[ E, G \text{ are much easier to calculate than fidelity } F. \]

(See also P. E. M. F. Mendonca, *et al.*, arXiv:0806.1150)
Conclusions

- Proposed bounds share with fidelity its main features (they are bounded, symmetric, unitary invariant and concave).
- Super–fidelity $G$ can be used in place of fidelity $F$ for small systems or when at least one of the states is pure enough.
- Sub- and super-fidelity can be (in principle) measured in laboratory.
- It is easy to calculate them using standard computer algebra systems.
References


Thank you.